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Structure of the Chiral Scalar Superfield in Ten Dimensions

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Abstract

We describe the tensors and spinor-tensors included in the θ-expansion of the ten-dimension chiral scalar superfield. The product decompositions of all the irreducible structures with θ ar the θ^2 tensor are provided as a first step towards the obtention of a full tensor calculus for th superfield.

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I Introduction

The field structure of higher dimensional supergravities as well as of $N \geq 3$ extended supergra ities is still an open problem. It is an old problem whose general solution was deemed impossib for a while due to some "no-go theorems" [1] establishing the impossibility of writing quadrat Lagrangians for the linearized (free) theory. The underlying problem was the so-called "se duality counting paradox" [2] which was subsequently resolved [3] by the discovery of the fa that the Lagrangian for the linear theory is not quadratic when is dealing with fields having self-dual field strength.

In particular one would really like to know the auxiliary field structure of 10-dimension supergravity [4], a theory unaffected by the above mentioned no-go theorems, due to its relevan for string theory applications.

Traditionally the auxiliary field structures for supergravities that are known have always be found in a rather *ad hoc* manner by counting degrees of freedom and trying to add suitable ne fields in order to match the bosonic and fermionic degrees of freedom off-shell [5]. It was on later, after the answer was known, that more systematic ways of deriving the result were foun However, for the more complicated theories the auxiliary field structure becomes so complex th it has been impossible to guess. Complicating matters further is the above-mentioned self-duali counting paradox, and we are finally bound to use a systematic approach to solve the problem

A fruitful approach in 4 dimensions is the use of the superconformal framework in whi the different Poincaré supergravities correspond to using different compensators to fix the ext degree of freedom [6]. However, while the super-Poincaré algebra remains essentially the same higher dimensions, the same is not true for the superconformal one which acquires a multitu of new generators [7], which complicates enormously this gauge-fixing procedure. In fact, ev though the complete off-shell structure of ten-dimensional conformal supergravity was obtain long ago in [8], a satisfactory off-shell Poincaré version is still lacking (see [9, 10]).

In ref. [10] it was proposed a linearized off-shell 10-dimensional supergravity adding to t conformal supergravity multiplet a set of 2 full-fledged chiral scalar superfields. However this in all likelihood a reducible version since each chiral scalar superfield contains 3 irreducible piec [11]. Furthermore, the tensorial structure and transformation rules of the component fields w not provided, even at the linearized level.

A second more promising approach is the irreducible superfield method, which has be successfully used in the $N = 1$ [12] and $N = 2$ [13] cases. In working with superfields [1 one is automatically assured that the numbers of fermionic and bosonic degrees of freedom w match, but general superfields are usually objects too large to handle, containing many mo fields that one is interested in, especially in higher dimensions (though some interesting fou dimensional results have been obtained using unconstrained superfields in the so called harmon superspace approach [14]). That is why the importance of irreducible superfields, which are mu simpler objects satisfying additional supersymmetric constraints. These subsidiary conditions a usually differential equations involving the superspace covariant derivatives, and can be obtain by applying appropriate projection operators for the corresponding eigenvalues of the Casimi [12]. The Casimir operators for the super-Poincaré algebras in all dimensions are known an they have been used to decompose the 11-dimensional [16] and 10-dimensional massive scal superfields. In the 10-dimensional case, there is an additional interesting complication, name that the lowest (quadratic) Casimir operator C_2 does not distinguish between the 3 irreducib

pieces since it has the same eigenvalue for the corresponding representation [11]. Therefo one would have to construct projection operators using the second lowest (quartic) Casim operator C_4, which does distinguish among those representations, but the resulting differenti equations are so complicated as to render the method impractical. However, this difficul was circumvented by resorting to the Cartan subalgebra in order to obtain simple differenti equations which were used to characterize the irreducible pieces of the massless and massi 10-dimensional scalar superfield in [17] and [18] respectively. The irreducible superfields we then obtained as expansions in Grassmann-Hermite polynomials, but the field components these non-covariant expressions remained to be sorted out, though in principle it can be done

In all this one final basic stumbling block remains though: while it is known from grou theory methods what are the fields contained in scalar superfield [19], it is not known in wh form they appear. In other words, while it is trivial to write the scalar superfield in multispin language:

$$\Phi(x,\theta) = \sum_{j=0}^{16} \chi_{\alpha_1 \ldots \alpha_j}(x) \theta^{\alpha_1} \ldots \theta^{\alpha_j}, \qquad (1.$$

it is a rather different proposition to extract the irreducible fields with their tensor (non-spinc indices out of the $\chi_{\alpha_1 \ldots \alpha_j}(x)$ fields. The latter is equivalent to decompose into irreducible pieces a the possible powers of the anticommuting variable θ^α, and that is what we will do in this pape The irreducible SO(10) representations contained in the corresponding powers of θ are reproduc in Table 1. The list is for increasing powers of one of the basic spinorial representations $[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}$ corresponding to the positive chirality projection $\theta^{(+)}$. For the negative chirality case $\theta^{(-)}$ o just needs to read Table 1 upside down. In either case the representations corresponding to t fields $\chi_{\alpha_1 \ldots \alpha_j}(x)$ are the same but with opposite chirality and duality when they apply. In oth words, the representations for the fields accompanying a certain power of $\theta^{(+)}$ are given by t same power of $\theta^{(-)}$ and viceversa.

j	$\theta^{\alpha_1}\ldots\theta^{\alpha_j}$	Dimension
0	$[0]$	1
1	$[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}]$	16
2	$[1\ 1\ 1]$	120
3	$[\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}]$	560
4	$[2\ 2] \oplus [2\ 1\ 1\ 1-1]$	$770 + 1050$
5	$[\frac{5}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}] \oplus [\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{-3}{2}]$	$3696 + 672$
6	$[3\ 1\ 1] \oplus [2\ 2\ 1\ 1-1]$	$4312 + 3696$
7	$[\frac{7}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}] \oplus [\frac{5}{2}\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{-1}{2}]$	$2640 + 8800$
8	$[4] \oplus [3\ 1\ 1\ 1] \oplus [2\ 2\ 2]$	$660 + 8085 + 4125$
9	$[\frac{7}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}] \oplus [\frac{5}{2}\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}]$	$2640 + 8800$
10	$[3\ 1\ 1] \oplus [2\ 2\ 1\ 1\ 1]$	$4312 + 3696$
11	$[\frac{5}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}] \oplus [\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}]$	$3696 + 672$
12	$[2\ 2] \oplus [2\ 1\ 1\ 1\ 1]$	$770 + 1050$
13	$[\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}]$	560
14	$[1\ 1\ 1]$	120
15	$[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2} - \frac{1}{2}]$	16
16	$[0]$	1

Table 1: Decomposition of the totally antisymmetrized Kronecker (wedge) powers of the bas spinor representation of SO(10), as given by their highest weights.

II Fierz Identity

The 10-dimensional Fierz identity for strictly anticommuting θ's can be put in a very simp
form

$$\bar{\theta}^{(\pm)}O_1\theta^{(\pm)}\bar{\theta}^{(\pm)}O_2\theta^{(\pm)} = \frac{1}{96}\bar{\theta}^{(\pm)}O_1\Pi^{(\pm)}\Gamma^{B_1B_2B_3}O_2\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{B_1B_2B_3}\theta^{(\pm)} \qquad (2.$$

where $\Pi^{(\pm)} = \frac{1}{2}(I \pm \Gamma_{(11)})$ are the Weyl projection operators (see Appendix A for our conventions
Then one obtains immediately the vanishing of the triple contraction:

$$\bar{\theta}^{(\pm)}\Gamma_{B_1B_2B_3}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{B_1B_2B_3}\theta^{(\pm)} = 0 \qquad (2.$$

since, in 10 dimensions, $\Gamma_{B_1B_2B_3}\Gamma^{C_1C_2C_3}\Gamma^{B_1B_2B_3} = -48\Gamma^{C_1C_2C_3}$. Likewise, using the properties
the Dirac algebra, it is relatively simple to show that the following double contraction vanishe

$$\bar{\theta}^{(\pm)}\Gamma_{AB_1B_2}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{B_1B_2C}\theta^{(\pm)} = \bar{\theta}^{(\pm)}\Gamma_A\Gamma_{B_1B_2}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{B_1B_2}\Gamma^C\theta^{(\pm)} = 0. \qquad (2.$$

For the single trace we get a non-trivial result:

$$\bar{\theta}^{(\pm)}\Gamma_{A_1A_2}\Gamma_B\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{C_1C_2}\Gamma^B\theta^{(\pm)} = 2\bar{\theta}^{(\pm)}\Gamma_{B[A_1}{}^{[C_1}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{A_2]}{}^{C_2]B}\theta^{(\pm)}. \qquad (2.$$

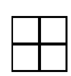
In particular, (2.4) implies the vanishing of the antisymmetric combination:

$$\bar{\theta}^{(\pm)}\Gamma_{[A_1A_2}\Gamma^B\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{C_1C_2]}\Gamma_B\theta^{(\pm)} = 0. \qquad (2.$$

In fact, (2.4) implies the more powerful and useful result

$$\bar{\theta}^{(\pm)}\Gamma_{[A_1A_2}\Gamma^B\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{A_3]}{}^C{}_B\theta^{(\pm)} = 0. \qquad (2.$$

Therefore we conclude that $\bar{\theta}^{(\pm)}\Gamma_{A_1A_2}\Gamma_B\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{C_1C_2}\Gamma^B\theta^{(\pm)}$ is a traceless tensor which contai
no antisymmetric parts of more than 2 indices, and must therefore correspond to the represe
tation

 or [2 2].

Finally we are ready to tackle the uncontracted product, and we obtain:

$$\begin{aligned}
&\frac{9}{8}\bar{\theta}^{(\pm)}\Gamma^{A_1A_2A_3}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{C_1C_2C_3}\theta^{(\pm)} = \\
&\quad \mp\frac{1}{32}\epsilon^{A_1A_2A_3D_1D_2D_3D_4D_5[C_1C_2}\bar{\theta}^{(\pm)}\Gamma^{C_3]}{}_{D_1D_2}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{D_3D_4D_5}\theta^{(\pm)} \\
&\quad -\frac{9}{8}\bar{\theta}^{(\pm)}\Gamma^{[A_1A_2[C_1}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{A_3]C_2C_3]}\theta^{(\pm)} \\
&\quad +\frac{9}{4}\eta^{[A_1[C_1}\bar{\theta}^{(\pm)}\Gamma^{A_2A_3]}{}_D\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{DC_2C_3]}\theta^{(\pm)} \qquad (2.
\end{aligned}$$

where one has to make use of the Dirac algebra and in particular

$$\Gamma^{A_1\ldots A_7} = \frac{1}{3!}\epsilon^{A_1\ldots A_7 B_1 B_2 B_3}\Gamma_{(11)}\Gamma_{B_1 B_2 B_3}. \quad (2.$$

Before we can make sense of Eq. (2.7), let us note that if we call:

$$X^{(\pm)C;D_1\ldots D_5} = \bar{\theta}^{(\pm)}\Gamma^{C[D_1 D_2}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{D_3 D_4 D_5]}\theta \quad (2.$$

we get

$$X^{(\pm)}_{[C_1;C_2C_3]}{}^{A_1A_2A_3} = \frac{1}{10}(\bar{\theta}^{(\pm)}\Gamma^{A_1A_2A_3}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{C_1C_2C_3}\theta^{(\pm)} - 3\bar{\theta}^{(\pm)}\Gamma^{[A_1A_2}{}_{[C_1}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{A_3]}{}_{C_2C_3]}\theta^{(\pm)}). \quad (2.1$$

$X^{(\pm)}$ is clearly traceless by virtue of (2.5) and trivially satisfies

$$X^{(\pm)[A;B_1\ldots B_5]} = 0. \quad (2.1$$

And, since $X^{(\pm)}$ has five totally antisymmetric indices, it is a good candidate for the oth
irreducible piece of the θ^4 sector. This will be confirmed shortly. Then we can rewrite (2.7) a

$$\begin{aligned}\bar{\theta}^{(\pm)}\Gamma^{A_1A_2A_3}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma^{C_1C_2C_3}\theta^{(\pm)} = &\\ \mp\frac{1}{48}\epsilon^{A_1A_2A_3D_1D_2D_3D_4D_5[C_1C_2}\bar{\theta}^{(\pm)C_3]}{}_{D_1D_2}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{D_3D_4D_5}\theta^{(\pm)} + &\\ +\frac{5}{2}X^{(\pm)[A_1;A_2A_3]C_1C_2C_3} &\\ +\frac{3}{2}\bar{\theta}^{(\pm)}\Gamma_B{}^{[A_2A_3}\theta^{(\pm)}\eta^{A_1][C_1}\bar{\theta}^{(\pm)}\Gamma^{C_2C_3]B}\theta^{(\pm)}. &\quad (2.1\end{aligned}$$

This equation implies the (anti-) self-duality of $X^{(\pm)A;B_1\ldots B_5}$:

$$\begin{aligned}X^{(\pm)A;B_1\ldots B_5} &= \mp\frac{1}{5!}\epsilon^{B_1\ldots B_5D_1\ldots D_5}X^{(\pm)A;}{}_{D_1\ldots D_5}\\ X^{(\pm)A;}{}_{B_1\ldots B_5} &= \pm\frac{1}{5!}\epsilon_{B_1\ldots B_5D_1\ldots D_5}X^{(\pm)A;D_1\ldots D_5} \quad (2.1\end{aligned}$$

thus confirming that it is the missing irreducible piece from the θ^4 sector.

Therefore, the basic identity (2.12) gives the decomposition of the general θ^4 tensor in i
reducible pieces. It is the basic identity from which all the higher order decompositions mu
necessarily follow by appropriate iterative use of it.

In the remainder of the paper we are going to concentrate only on the positive chirality ca
$\theta^{(+)}$. To obtain the corresponding results for $\theta^{(-)}$ one just has to remember that all the chirali
and duality properties are reversed.

III θ^6 Decompositions

In order to simplify notation let us call

$$M^{ABC} = \bar{\theta}^{(+)}\Gamma^{ABC}\theta^{(+)}. \quad (3.$$

Also in the remainder of the paper we are going to use the following letter convention: *u contracted indices labeled by the same letter with different subindex are understood to be a tisymmetrized except if the letter involved is S or X in which case they are understood to symmetrized*. For instance:

$$F^{CA_1A_2A_3}G^{A_4A_5D} \equiv F^{C[A_1A_2A_3}G^{A_4A_5]D}$$

$$N^{CDS_1S_2}{}_{X_1}P^{S_3AB}{}_{X_2} \equiv N^{CD(S_1S_2}{}_{(X_1}P^{S_3)AB}{}_{X_2)} \quad (3.$$

where the square and round brackets are the by now standard notations denoting normalized tot antisymmetrization and symmetrization respectively. This notation will dramatically reduce t need for brackets which would make some formulae otherwise practically impossible to write.

Then, Eq. (2.12) becomes:

$$\begin{aligned} M^{A_1A_2A_3}M^{B_1B_2B_3} &= \\ &\frac{5}{2}\left(M^{A_1[A_2A_3}M^{B_1B_2B_3]} - \frac{1}{5!}\epsilon^{A_1A_2A_3B_1B_2D_1\dots D_5}M^{B_3}{}_{D_1D_2}M_{D_3D_4D_5}\right) \\ &+\frac{3}{2}\eta^{A_1B_1}M^{A_2A_3}{}_D M^{B_2B_3D}. \end{aligned} \quad (3.$$

Eq. (3.3) is equivalent to the following two statements:

$$M^{CA_1A_2}M^{A_3A_4A_5} = -\frac{1}{5!}\epsilon^{A_1\dots A_5B_1\dots B_5}M^C{}_{B_1B_2}M_{B_3B_4B_5} \quad (3.$$

$$M^{A_1A_2A_3}M^{B_1B_2B_3} = 5M^{A_1[A_2A_3}M^{B_1B_2B_3]} + \frac{3}{2}\eta^{A_1B_1}M^{A_2A_3}{}_D M^{B_2B_3D}. \quad (3.$$

Eqs. (3.3) or (3.5) clearly give the decomposition of $M^{A_1A_2A_3}M^{B_1B_2B_3}$ into its irreducible part the anti-selfdual $[2111-1]$ piece:

$$\mathcal{M}_4^{A;B_1\dots B_5} = M^{AB_1B_2}M^{B_3B_4B_5} \quad (3.$$

and the $[22]$ piece:

$$\mathcal{M}_4^{A_1A_2;B_1B_2} = M^{A_1A_2}{}_E M^{B_1B_2E}. \quad (3.$$

From their definitions and the results of this and the previous section, we get the followi properties:

$$\mathcal{M}_4^{[A;B_1\dots B_5]} = 0 \qquad \mathcal{M}_4{}_E{}^{;EB_1\dots B_4} = 0$$

$$\mathcal{M}_4^{A;B_1\dots B_5} = -\frac{1}{5!}\epsilon^{B_1\dots B_5D_1\dots D_5}\mathcal{M}_4{}^{A;}{}_{D_1\dots D_5} \quad (3.$$

and

$$\mathcal{M}_4^{A_1A_2;B_1B_2} = \mathcal{M}_4^{B_1B_2;A_1A_2} \qquad \mathcal{M}_{4\ E}^{\ \ A;EB} = 0$$
$$\mathcal{M}_4^{A[B;CD]} = 0. \qquad (3.$$

In order to decompose the next product $M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}$ one can proceed to itera (3.3) for the different binary products. After several iterations and a lot of algebra it is possib to obtain the following decomposition:

$$\begin{aligned}
&M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} = \\
&\quad \mathcal{S}(A,B,C)\Big\{18\eta^{B_1C_1}M^{[A_1A_2A_3}M^{B_2C_2]}{}_D M^{B_3C_3D} + \frac{18}{5}\eta^{A_1C_1}\eta^{B_1C_2}M^{C_3}{}_{DE}M^{A_2A_3D}M^{B_2B_3E} \\
&\quad -\frac{9}{5}\eta^{B_1C_1}\eta^{B_2C_2}M^{C_3}{}_{DE}M^{B_3DA_1}M^{A_2A_3E}\Big\} \\
&\quad +\frac{1}{20}\epsilon^{B_1B_2B_3A_1A_2C_1C_2D_1D_2D_3}M^{A_3E_1}{}_{D_1}M_{D_2D_3}{}^{E_2}M_{E_1E_2}{}^{C_3} \qquad (3.1
\end{aligned}$$

where $\mathcal{S}(A,B,C)$ is the normalized operator that fully symmetrizes on the letters A,B,C. T last term in (3.10) is automatically symmetric upon interchange of these three letters, as can easily proven by using the fact that a complete antisymmetrization of 11 indices must necessari vanish.

In deriving (3.10) one has to make use of many identities (see Appendix A) which are al consequences of (3.3), specially

$$M^A{}_{DE}M^{BEF}M^C{}_F{}^D = 0 \qquad (3.1$$

which follows almost immediately from (2.6) and (2.3). Eq. (3.11) means that *all triple contra tions of M^3 vanish*, as it should be since there are no objects with 3 indices in the θ^6 sector.

The amount of effort required to obtain (3.10) by iteration of (3.3) makes it clear that alternative way is needed if one hopes to decompose all the higher order products. Neverthele it illustrates the fact that all the necessary product decompositions are direct consequences the Fierz identity (2.12).

There is a much simpler way to obtain the decomposition (3.10), by systematically removi traces (since the irreducible pieces are traceless) and using the appropriate Young projectors the traceless parts. This is possible because we already know beforehand what are the irreducib representations involved (see Table 1).

Let us begin by removing all the traces from the object:

$$\begin{aligned}
M^{A_1A_2A_3}M_D{}^{B_1B_2}M^{DC_1C_2} =\ & \mathit{Traceless}\ (M^{A_1A_2A_3}M_D{}^{B_1B_2}M^{DC_1C_2}) \\
&+\frac{2}{5}(2\eta^{A_1B_1}M^{EA_2A_3}M_{DE}{}^{B_2}M^{DC_1C_2} + 2\eta^{A_1C_1}M^{EA_2A_3}M^{DB_1B_2}M_{DE}{}^{C_2} \\
&+\eta^{B_1C_1}M^{EA_2A_3}M_{DE}{}^{B_2}M^{DC_2A_1}) \qquad (3.1
\end{aligned}$$

Next we decompose $\mathit{Traceless}\ (M^{A_1A_2A_3}M_D{}^{B_1B_2}M^{DC_1C_2})$ using the Young projectors correspon ing to the representation (see Table 1) whose construction is detailed in Appendix C:

$$
\begin{aligned}
&Traceless\,(M^{A_1A_2A_3}M_D{}^{B_1B_2}M^{DC_1C_2}) = \\
&= Y\left(\yng(2,2,1,1,1)\right) M^{A_1A_2A_3}M^{B_1B_2D}M^{C_1C_2}{}_D \\
&= \frac{2}{3}(M^{[A_1A_2A_3}M^{B_1B_2]}{}_D M^{C_1C_2D} + M^{[A_1A_2A_3}M^{C_1C_2]}{}_D M^{B_1B_2D} \\
&+2M^{[A_1A_2A_3}M^{B_1C_1]}{}_D M^{B_2C_2D}). \qquad \text{(3.1}
\end{aligned}
$$

Now we do the same for the uncontracted product $M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}$, first remo the traces:

$$
\begin{aligned}
M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} &= Traceless\,(M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}) \\
&+\frac{9}{5}\mathcal{S}(A,B,C)\Big\{\eta^{A_1B_1}\Big[\frac{3}{2}M_D{}^{A_2A_3}M^{DB_2B_3}M^{C_1C_2C_3} - M_D{}^{A_2A_3}M^{C_1B_2B_3}M^{DC_2C_3} \\
&+2M_D{}^{A_2A_3}M^{C_1C_2B_2}M^{DB_3C_3} + 2M_D{}^{A_2B_2}M^{DC_1C_2}M^{A_3B_3C_3}\Big]\Big\}. \qquad \text{(3.1}
\end{aligned}
$$

Using some of the identities in Appendix A and the decomposition (3.12)-(3.13) we get

$$
\begin{aligned}
M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} &= Traceless\,(M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}) \\
&+9\mathcal{S}(A,B,C)\Big\{2\eta^{A_1B_1}M^{[C_1C_2C_3}M^{A_2B_2]}{}_D M^{A_3B_3D} + \frac{2}{5}\eta^{A_1B_1}\eta^{C_1B_2}M^{B_3}{}_{DE}M^{A_2A_3D}M^{C_2C_3E} \\
&-\frac{1}{5}\eta^{A_1B_1}\eta^{A_2B_2}M^{B_3}{}_{DE}M^{C_1C_2D}M^{B_3C_3E}\Big\}. \qquad \text{(3.1}
\end{aligned}
$$

To obtain the traceless part in (3.15), we apply the Young projector corresponding to the repr sentation $\yng(3,1,1,1,1,1,1)$ ($\equiv \yng(3,1)$ for $SO(10)$)

$$
\begin{aligned}
&Traceless\,(M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}) = \\
&= Y\left(\yng(3,1,1,1,1,1,1)\right) M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} \\
&= 21M^{B_1[B_2B_3}M^{A_1A_2A_3}M^{C_2C_3]C_1} \\
&= -\frac{1}{20}\epsilon^{A_1A_2A_3B_1B_2C_1C_2E_1E_2E_3}M^{FDB_3}M_{DE_1E_2}M_{E_3F}{}^{C_3} \qquad \text{(3.1}
\end{aligned}
$$

where the last equality follows from the anti-selfduality of $M^{A[B_1B_2}M^{B_3B_4B_5]}$ by rotating indice and explicitly displays the aforementioned equivalence of $SO(10)$ representations.

Eq. (3.16) together with (3.15) reproduces for us the decomposition (3.10). We will del the study of the irreducible pieces of the θ^6 sector until the next section.

IV Irreducible Bosonic Structures

The difficulty in proceeding along the lines of the previous section is that one needs to kno beforehand what are the irreducible pieces of the higher θ powers in order to decompose t products into irreducible pieces. That is why we are now going to proceed *backwards*, startin from the scalar corresponding to θ^{16} and come down from there.

To construct the above scalar we first notice that it is easy to identify the totally symmet tensor of θ^8 sector corresponding to the representation [4]:

$$\mathcal{M}_8^{ABCD} = M^A{}_E{}^F M^B{}_F{}^G M^C{}_G{}^H M^D{}_H{}^E. \qquad (4.$$

It is obviously traceless (see (3.11)and cyclically symmetric:

$$\mathcal{M}_8^{ABCD} = \mathcal{M}_8^{DABC} \qquad (4.$$

and the antisymmetrization of any two neighboring indices vanishes

$$\begin{aligned}\mathcal{M}_8^{[AB]CD} &= M^{[A}{}_{EF} M^{B]FG} M^C{}_G{}^H M^D{}_H{}^E \\ &= -\frac{1}{2} M^{BA}{}_F M_E{}^{FG} M^C{}_{GH} M^{DHE} \\ &= \frac{1}{4} M^{BA}{}_F M_{HE}{}^G M^C{}_G{}^F M^{DHE} = 0 \qquad (4.\end{aligned}$$

where we have twice made use of (2.6) and then (2.3). Thus

$$\mathcal{M}_8^{ABCD} = \mathcal{M}_8^{BACD}. \qquad (4.$$

Properties (4.2) and (4.4) imply that $\mathcal{M}_8^{ABCD}$ is completely symmetric in all four indices.

$\underline{\theta^{16}}$.

The scalar we are looking for is the square of (4.1)

$$\begin{aligned}\mathcal{M}_{16} &= \mathcal{M}_8^{S_1S_2S_3S_4} \mathcal{M}_{8\,S_1S_2S_3S_4} \\ &= M^{S_1}{}_{E_1E_2} M^{S_2E_2F_1} M^{S_3}{}_{F_1F_2} M^{S_4F_2E_1} M_{S_1}{}^{G_1G_2} M_{S_2G_2H_1} M_{S_3}{}^{H_1H_2} M_{S_4H_2G_1} \qquad (4.\end{aligned}$$

where all the M-factors are equivalent.

$\underline{\theta^{14}}$.

Since all the factors in (4.5) are equivalent, there is only one possible expression to be obtain by removing any one of them and that must be our irreducible piece:

$$\begin{aligned}\mathcal{M}^{ABC} &= M^{S_1}{}_{E_1E_2} M^{S_2E_2F_1} M^{S_3}{}_{F_1F_2} M^{AF_2E_1} M_{S_1}{}^{BG} M_{S_2GH} M_{S_3}{}^{HC} \\ &= \mathcal{M}_8^{S_1S_2S_3A} M_{S_1}{}^{BG} M_{S_2GH} M_{S_3}{}^{HC} \qquad (4.\end{aligned}$$

which is obviously antisymmetric in B, C:

$$\mathcal{M}^{ABC} = -\mathcal{M}^{ACB} \qquad (4.$$

but must be totally antisymmetric because it must belong to $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array} \equiv [1\ 1\ 1]$. In order to pro
this, we first put it in a more appealing form using the symmetry of the $\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array}$ part as well
(2.6):

$$\mathcal{M}^{ABC} = M^{AS_1D_1} M^{S_2}{}_{D_1D_2} M^{ED_2F_1} M^{B}{}_{F_1F_2} M_{S_1}{}^{F_2G_1} M_{S_2G_1G_2} M^{C}{}_{E}{}^{G_2}. \qquad (4.$$

Then, reordering factors and using (2.6) once more we obtain

$$\begin{aligned} \mathcal{M}^{ABC} &= M^{BS_1D_1} M^{S_2}{}_{D_1D_2} M^{ED_2F_1} M^{C}{}_{F_1F_2} M_{S_1}{}^{F_2G_1} M_{S_2G_1G_2} M^{A}{}_{E}{}^{G_2} \\ &= \mathcal{M}^{BCA}. \end{aligned} \qquad (4.$$

Properties (4.7) and (4.9) imply that $\mathcal{M}^{ABC}$ is completely antisymmetric in all 3 indice
From (4.6) and (4.5) we note that

$$\mathcal{M}^{A_1A_2A_3} M_{A_1A_2A_3} = -\mathcal{M}_{16} \qquad (4.1$$

and therefore we have the product decomposition

$$\mathcal{M}^{A_1A_2A_3} M^{B_1B_2B_3} = -\frac{1}{120} \eta^{A_1B_1} \eta^{A_2B_2} \eta^{A_3B_3} \mathcal{M}_{16}. \qquad (4.1$$

$\underline{\theta^{12}}$.

Not all the factors in (4.6) are equivalent, so now we get two possible structures by removi
one factor from $\mathcal{M}^{ABC}$. One is:

$$\hat{\mathcal{M}}_{12}{}^{AB,CD} = M^{AS_1}{}_{D_1} M^{S_2D_1D_2} M^{C}{}_{D_2E} M^{DEF_1} M_{S_1F_1F_2} M_{S_2}{}^{F_2B} \qquad (4.1$$

which is clearly traceless and, by virtue of (2.6), (2.3), has the symmetry properties:

$$\hat{\mathcal{M}}_{12}{}^{AB,CD} = \hat{\mathcal{M}}_{12}{}^{BA,CD} = \hat{\mathcal{M}}_{12}{}^{AB,DC} = \hat{\mathcal{M}}_{12}{}^{BA,DC}. \qquad (4.1$$

By using (2.6) in a different way we can also derive

$$\hat{\mathcal{M}}_{12}{}^{AB,CD} + \hat{\mathcal{M}}_{12}{}^{AC,DB} + \hat{\mathcal{M}}_{12}^{CB,AD} = 0 \qquad (4.1$$

$$\hat{\mathcal{M}}_{12}{}^{AB,CD} + \hat{\mathcal{M}}_{12}{}^{DB,AC} + \hat{\mathcal{M}}_{12}^{AD,CB} = 0. \qquad (4.1$$

Combining (4.14) with (4.13) we get

$$\hat{\mathcal{M}}_{12}{}^{A[B,C]D} + \hat{\mathcal{M}}_{12}{}^{D[B,C]A} = 0 \qquad (4.1$$

while combining (4.14) and (4.15),

$$\hat{\mathcal{M}}_{12}{}^{AB,CD} = \hat{\mathcal{M}}_{12}{}^{CD,AB}. \qquad (4.1$$

Once we have obtained (4.17) we see that (4.14) and (4.15) simply mean:

$$\hat{\mathcal{M}}_{12}{}^{A(B,CD)} = 0. \qquad (4.1$$

Eq. (4.16) tells us that antisymmetrizing on two indices on opposite sides of the comm automatically makes the other pair also antisymmetric. Thus we recognize the object th displays the symmetry of the Young pattern $\boxplus$:

$$\mathcal{M}_{12}{}^{A_1A_2;B_1B_2} = \hat{\mathcal{M}}_{12}{}^{A_1B_1,B_2A_2} = \hat{\mathcal{M}}_{12}{}^{A_1B_1,A_2B_2}. \qquad (4.1$$

However it is interesting to note for reference, the more interesting properties of the $\hat{\mathcal{M}}_{12}$ tenso From the definition (4.19) it is clear that $\hat{\mathcal{M}}_{12}^{A_1A_2,B_1B_2}$ is traceless and that it satisfies:

$$\mathcal{M}_{12}{}^{A[B;CD]} = 0. \qquad (4.2$$

Thus it has the same properties as the tensor $\mathcal{M}_{12}{}^{A_1A_2;B_1B_2}$ except for nilpotency.

Even though $\hat{\mathcal{M}}_{12}$ and $\mathcal{M}_{12}$ have apparently different symmetry properties they both ha the same number of degrees of freedom, 770, i.e. the dimension of the irrep. [22] of $SO(10)$, a they both can be expressed in terms of the other. The inverse of (4.19) is

$$\hat{\mathcal{M}}_{12}{}^{AB,CD} = \frac{2}{3}(\mathcal{M}_{12}{}^{AD;BC} + \mathcal{M}_{12}{}^{BD;AC}) \qquad (4.2$$

as can be easily seen by using (4.18).

From (4.8) and (4.12) we see that

$$\hat{\mathcal{M}}_{12}^{AE,BF} M^{C}{}_{FE} = \mathcal{M}_{12}{}^{AB;EF} M^{C}{}_{FE} = \mathcal{M}^{ABC} \qquad (4.2$$

and then we have for the decomposition of the single contraction:

$$\hat{\mathcal{M}}_{12}{}^{S_1S_2,XE} M_{A_1A_2E} = \frac{1}{7}\Big(3\delta^{S_1}_{A_1}\mathcal{M}^{S_2X}{}_{A_2} - \frac{1}{3}\eta^{XS_1}\mathcal{M}^{S_2}{}_{A_1A_2} + \frac{1}{3}\eta^{S_1S_2}\mathcal{M}^{X}{}_{A_2A_1}\Big). \qquad (4.2$$

Eq. (4.23) is easily obtained since it must have that general form and the coefficients are giv by the traces of the left-hand side, either zero or (4.22). For the other object we have

$$\mathcal{M}_{12}{}^{B_1B_2;CE} M_E{}^{A_1A_2} = \frac{1}{14}(3\eta^{A_1C}\mathcal{M}^{A_2B_1B_2} - 3\eta^{A_1B_1}\mathcal{M}^{A_2B_2C} + \eta^{CB_1}\mathcal{M}^{B_2A_1A_2}). \qquad (4.2$$

Using (4.23) and following the same procedure one derives for the full product

$$\begin{aligned}
\hat{\mathcal{M}}_{12}{}^{S_1S_2,X_1X_2} M^{A_1A_2A_3} \quad = \quad & \frac{3}{11\times 7}\Big[8\eta^{S_1A_1}\eta^{X_1A_2}\mathcal{M}^{S_2X_2A_3} \\
& -\eta^{S_1X_1}\eta^{S_2A_1}\mathcal{M}^{X_2A_2A_3} - \eta^{S_1X_1}\eta^{X_2A_1}\mathcal{M}^{S_2A_2A_3} \\
& +\eta^{X_1X_2}\eta^{S_1A_1}\mathcal{M}^{S_2A_2A_3} + \eta^{S_1S_2}\eta^{X_1A_1}\mathcal{M}^{X_2A_2A_3} \\
& -\frac{1}{9}\left(\eta^{X_1X_2}\eta^{S_1S_2} - \eta^{X_1S_1}\eta^{X_2S_2}\right)\mathcal{M}^{A_1A_2A_3}\Big] \qquad (4.2
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{M}_{12}{}^{B_1B_2;C_1C_2} M^{A_1A_2A_3} &= -\frac{6}{11\times 7}\Big(\eta^{A_1B_1}\eta^{A_2B_2}\mathcal{M}^{A_2C_1C_2} + 2\eta^{A_1B_1}\eta^{A_2C_2}\mathcal{M}^{A_3B_2C_2} \\
&+ \eta^{A_1C_1}\eta^{A_2C_2}\mathcal{M}^{A_3B_1B_2} - \frac{3}{4}\eta^{B_1C_1}\eta^{C_2A_1}\mathcal{M}^{A_2A_3B_2} \\
&- \frac{3}{4}\eta^{B_1C_1}\eta^{B_2A_1}\mathcal{M}^{A_2A_3C_2} + \frac{1}{12}\eta^{B_1C_1}\eta^{B_2C_3}\mathcal{M}^{A_1A_2A_3}\Big). \qquad (4.2
\end{aligned}
$$

If we remove a different factor from $\mathcal{M}^{ABC}$ we extract the new structure

$$
\hat{\mathcal{M}}_{12}{}^{XABY;E_1E_2} = M^X{}_{D_1}{}^{D_2} M^A{}_{D_2}{}^{D_3} M^F{}_{D_3}{}^{D_4} M^B{}_{D_4}{}^{D_5} M^Y{}_{D_5}{}^{D_1} M_F{}^{E_1E_2}. \qquad (4.2
$$

It has the obvious property

$$
\hat{\mathcal{M}}_{12}^{XABY;C_1C_2} = -\hat{\mathcal{M}}_{12}^{YBAX;C_1C_2} \qquad (4.2
$$

and by applying (2.6) it is also easy to prove

$$
\hat{\mathcal{M}}_{12}^{X[ABY;C_1C_2]} = \hat{\mathcal{M}}_{12}^{[A^X BY;C_1C_2]} \qquad (4.2
$$

which in turn implies:

$$
\hat{\mathcal{M}}_{12}^{[XABY;C_1C_2]} = 0. \qquad (4.3
$$

However, this object is not irreducible because it is not completely traceless, but rather h two non-vanishing traces:

$$
\begin{aligned}
\hat{\mathcal{M}}_{12\ E}{}^{ABY;EC} &= -\hat{\mathcal{M}}_{12}{}^{AC,BY} \\
\hat{\mathcal{M}}_{12}{}^{XAB}{}_E{}^{;EC} &= \hat{\mathcal{M}}_{12}{}^{BC,AX}. \qquad (4.3
\end{aligned}
$$

In order to decompose it one removes the traces and applies the appropriate Young projector

$$
\begin{aligned}
\hat{\mathcal{M}}_{12}{}^{XABY;C_1C_2} &= Traceless\,(\hat{\mathcal{M}}_{12}{}^{XABY;C_1C_2}) \\
&+\frac{1}{9\times 21}\Big\{-46(\eta^{XC_1}\hat{\mathcal{M}}_{12}^{AC_2,BY} - \eta^{YC_1}\hat{\mathcal{M}}_{12}{}^{BC_3,AX}) \\
&-3(\eta^{XB}\hat{\mathcal{M}}_{12}{}^{AC_1,YC_2} - \eta^{YA}\hat{\mathcal{M}}_{12}{}^{BC_1,XC_2}) \\
&-5(\eta^{XC_1}\hat{\mathcal{M}}_{12}{}^{BC_2,AY} - \eta^{YC_1}\hat{\mathcal{M}}_{12}{}^{AC_1,BX} + \eta^{AC_1}\hat{\mathcal{M}}_{12}{}^{BC_2,XY} \\
&-\eta^{BC_1}\hat{\mathcal{M}}_{12}{}^{AC_2,XY} - \eta^{AC_1}\hat{\mathcal{M}}_{12}{}^{SC_2,BY} + \eta^{BC_1}\hat{\mathcal{M}}_{12}{}^{YC_1,AX}) \\
&+2(\eta^{XA}\hat{\mathcal{M}}_{12}^{BC_1,YC_2} - \eta^{YB}\hat{\mathcal{M}}_{12}{}^{AC_1,XC_2} \\
&+4\eta^{XY}\hat{\mathcal{M}}_{12}{}^{AC_1,BC_2} - \eta^{AB}\hat{\mathcal{M}}_{12}{}^{XC_1,YX_2})\Big\}, \qquad (4.3
\end{aligned}
$$

$$
\begin{aligned}
Traceless\,(\hat{\mathcal{M}}_{12}{}^{XABY;C_1C_2}) &= Y\left(\ydiagram{2,1,1,1}\right)\hat{\mathcal{M}}_{12}{}^{XABY;C_1C_2} \\
&= \frac{5}{6}(\hat{\mathcal{M}}_{12}{}^{X[ABY;C_2C_3]} + \hat{\mathcal{M}}_{12}{}^{A[BYX;C_1C_2]} \\
&\quad + \hat{\mathcal{M}}_{12}{}^{B[YXA;C_1C_2]} + \hat{\mathcal{M}}_{12}{}^{Y[XAB;C_1C_2]}) \qquad (4.3
\end{aligned}
$$

From (4.33) it is apparent that the second irreducible structure is

$$
\begin{aligned}
\mathcal{M}_{12}^{B;A_1\ldots A_5} &= \hat{\mathcal{M}}_{12}^{BA_1A_2A_3;A_4A_5} \\
&= M^B{}_{D_1}{}^{D_2}M^F{}_{D_2}{}^{D_3}M^{A_1}{}_{D_3}{}^{D_4}M^{A_2}{}_{D_4}{}^{D_5}M^{A_3}{}_{D_5}{}^{D_1}M_F{}^{A_4A_5}, \qquad (4.3
\end{aligned}
$$

whose tracelessness is confirmed by (3.11). Eq. (4.30) implies the property

$$
\mathcal{M}_{12}^{[B;A_1\ldots A_5]} = 0 \qquad (4.3
$$

and in Appendix A we prove the duality property

$$
\mathcal{M}_{12}^{B;A_1\ldots A_5} = \frac{1}{5!}\epsilon^{A_1\ldots A_5E_1\ldots E_5}\mathcal{M}_{12}^{B;}{}_{E_1\ldots E_5} \qquad (4.3
$$

which is opposite to the one satisfied by $\mathcal{M}_4{}^{B;A_1\ldots A_5}$. The definitions (4.34), (4.6) give the resu for the triple contractions

$$
\begin{aligned}
\mathcal{M}_{12}^{B;A_1A_2E_1E_2E_3}M_{E_1E_2E_3} &= -\frac{1}{5}\mathcal{M}^{BA_1A_2} \\
\mathcal{M}_{12}^{E_1;E_2E_3A_1A_2A_3}M_{E_1E_2E_3} &= -\frac{1}{5}\mathcal{M}^{A_1A_2A_3}. \qquad (4.3
\end{aligned}
$$

and by simple detracing,

$$
\begin{aligned}
\mathcal{M}_{12}{}^{B;A_1A_2A_3E_1E_2}M_{CE_1E_2} &= -\frac{1}{70}(\delta^B_C\mathcal{M}^{A_1A_2A_3} - \eta^{BA_1}\mathcal{M}_C^{A_2A_3} + 5\delta^{A_1}_C\mathcal{M}^{BA_2A_3}) \\
\mathcal{M}_{12}^{E_1;E_2A_1\ldots A_1}M_{CE_1E_2} &= -\frac{4}{35}\delta^{A_1}_C\mathcal{M}^{A_2A_3A_4}. \qquad (4.3
\end{aligned}
$$

From (4.38) and Young-projecting

$$
\begin{aligned}
\mathcal{M}_{12}{}^{B;A_1\ldots A_4E}M_{C_1C_2E} &= Traceless\,(\mathcal{M}_{12}{}^{B;A_1\ldots A_4E}M_{C_1C_2E}) \\
&+ \frac{2}{3\times 35}(\delta^B_{C_1}\mathcal{M}^{A_2A_3A_4} - 2\delta^{A_1}_{C_1}\delta^{A_2}_{C_2}\mathcal{M}^{BA_3A_4} - \eta^{BA_1}\delta^{A_2}_{C_1}\mathcal{M}_{C_2}{}^{A_3A_4}) \qquad (4.3
\end{aligned}
$$

$$
\begin{aligned}
Traceless\,(\mathcal{M}_{12}{}^{B;A_1\ldots A_4E}M^{C_1C_2}{}_E) &= Y\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline\end{array}\right)\mathcal{M}_{12}{}^{B;A_1\ldots A_4E}M^{C_1C_2}{}_E \\
&= \mathcal{M}_{12}{}^{[B;A_1\ldots A_4}{}_E M^{C_1C_2]E} = \frac{1}{5}\mathcal{M}_{12}{}^{E;[BA_1\ldots A_4}M^{C_1C_2]} \\
&= -\frac{4}{5}\frac{1}{7!}\epsilon^{BA_1\ldots A_4C_1C_2E_1E_2E_3}\mathcal{M}_{E_1E_2E_3}.
\end{aligned}
$$
(4.4

Eqs. (4.39), (4.40) and (4.35) then give

$$
\mathcal{M}_{12}^{E;A_1\ldots A_5}M^{C_1C_2}{}_E = -\frac{2}{21}\Big(\frac{1}{5!}\epsilon^{A_1\ldots A_5C_1C_2E_1E_2E_3}\mathcal{M}_{E_1E_2E_3} + \eta^{A_1C_1}\eta^{A_2C_2}\mathcal{M}^{A_3A_4A_5}\Big).
$$
(4.4

Finally for the full product

$$
\begin{aligned}
\mathcal{M}_{12}{}^{B;A_1\ldots A_5}M^{C_1C_2C_3} &= -\frac{2}{7!}\{\eta^{BC_1}\epsilon^{A_1\ldots A_5C_2C_3E_1E_2E_3} - \eta^{BA_1}\epsilon^{A_2\ldots A_5C_1C_2C_3E_1E_2E_3} \\
&\quad +\eta^{C_1A_1}\epsilon^{BA_2\ldots A_5C_2C_3E_1E_2E_3}\}\mathcal{M}_{E_1E_2E_3} \\
&- \frac{1}{35}\Big[\eta^{BC_1}\eta^{A_1C_2}\eta^{A_2C_3}\mathcal{M}^{A_3A_4A_5} + \eta^{A_1C_1}\eta^{A_2C_2}\eta^{A_3C_3}\mathcal{M}^{BA_4A_5} \\
&- \eta^{BA_1}\eta^{A_2C_1}\eta^{A_3C_2}\mathcal{M}^{A_4A_5C_3}\Big].
\end{aligned}
$$
(4.4

$\underline{\theta^{10}}$.

The first structure we encounter by removing a factor from $\hat{\mathcal{M}}_{12}{}^{AB,CD}$ is

$$
\hat{\mathcal{M}}_{10}{}^{S_1S_2S_3;A_1A_2} = \mathcal{M}_8^{S_1S_2S_3E}M^{A_1A_2}{}_E
$$
(4.4

whose symmetry properties are manifest. Its tracelessness follows from these symmetries a
from the tracelessness of $\mathcal{M}_8{}^{S_1S_2S_3S_4}$. The object in (4.43) also satisfies

$$
\hat{\mathcal{M}}_{10}{}^{(S_1S_2S_3;A)B} = 0,
$$
(4.4

and its product decompositions can be derived as before and we just list them:

$$
\hat{\mathcal{M}}_{10}{}^{EDA;BF}M^C{}_{EF} = -\hat{\mathcal{M}}_{12}{}^{AD,BC}
$$

$$
\hat{\mathcal{M}}_{10}{}^{ABD;EF}M^C{}_{EF} = \hat{\mathcal{M}}_{10}{}^{EFA;BD}M^C{}_{EF} = 0
$$

$$
\hat{\mathcal{M}}_{10}{}^{S_1S_2S_3;AE}M^{B_1B_2}{}_E = \frac{4}{7}\eta^{S_1B_1}\hat{\mathcal{M}}_{12}{}^{S_2S_3,AB_2} - \frac{2}{21}\eta^{S_1S_2}\hat{\mathcal{M}}_{12}{}^{S_3B_1,AB_2}
$$

$$
\begin{aligned}
\hat{\mathcal{M}}_{10}{}^{S_1S_2E;A_1A_2}M^{B_1B_2}{}_E &= -\frac{10}{3}\mathcal{M}_{12}{}^{S_1;S_2A_1A_2B_1B_2} \\
&+ \frac{2}{63}\Big[\frac{10}{3}(\eta^{S_1B_1}\hat{\mathcal{M}}_{12}{}^{A_1B_2,A_2S_2} + \eta^{S_1A_1}\hat{\mathcal{M}}_{12}{}^{A_2B_1,B_2S_2}) \\
&+ 17\eta^{A_1B_1}\hat{\mathcal{M}}_{12}{}^{S_1S_2,A_2B_2} + \frac{2}{3}\eta^{S_1S_2}\hat{\mathcal{M}}_{12}{}^{A_1B_1,A_2B_2}\Big] \\
\hat{\mathcal{M}}_{10}{}^{S_1S_2S_3;A_1A_2}M^{B_1B_2B_3} &= -\frac{15}{11}\Big\{\frac{18}{7}\eta^{S_1B_1}\mathcal{M}_{12}{}^{S_2;S_3A_1A_2B_2B_3} - \eta^{S_1S_2}\mathcal{M}_{12}{}^{B_1;S_3A_1A_2B_2B_3}\Big\} \\
&+ \frac{4}{7}(\eta^{S_1A_1}\mathcal{M}_{12}{}^{S_2;S_3A_2B_1B_2B_3} - \eta^{S_1S_2}\mathcal{M}_{12}{}^{A_1;S_3A_2B_1B_2B_3}) \\
&+ \frac{2}{35}\Big[2\eta^{S_1B_1}\eta^{S_2A_1}\hat{\mathcal{M}}_{12}{}^{A_2B_2,B_3S_3} + 9\eta^{S_1B_1}\eta^{B_2A_1}\hat{\mathcal{M}}_{12}{}^{S_2S_3,A_2B_3}\Big] \\
&- \frac{1}{21}\Big[2\eta^{S_1S_2}\eta^{A_1B_1}\hat{\mathcal{M}}_{12}{}^{A_2B_2,B_3S_3} - \frac{1}{5}\eta^{S_1S_2}\eta^{S_3B_1}\hat{\mathcal{M}}_{12}{}^{A_1B_2,A_2B_3}\Big].
\end{aligned}
$$

(4.4

However, the symmetry properties of the tensor $\hat{\mathcal{M}}_{10}{}^{S_1S_2S_3;A_1A_2}$ are not the ones of the Your pattern $\yng(3,1)$ as it is conventionally understood, but it is easy to construct a new tensor whic corresponds to $\yng(3,1)$:

$$\mathcal{M}_{10}{}^{S_1S_2;A_1A_2A_3} = \hat{\mathcal{M}}_{10}{}^{S_1S_2[A_1;A_2A_3]}. \quad (4.4$$

But, just like we had in the θ^{12} case, both of these objects are equivalent, both are irreducib and carry the same number of degrees of freedom (4312) and they can be expressed in terms each other. The inverse of (4.46) is:

$$\hat{\mathcal{M}}_{10}{}^{S_1S_2B;A_1A_2} = \frac{3}{5}(\mathcal{M}_{10}{}^{S_1S_2;BA_1A_2} + 2\mathcal{M}_{10}{}^{S_1B;S_2A_1A_2}). \quad (4.4$$

From the definition (4.46) we get the property

$$\mathcal{M}_{10}{}^{S[B;A_1A_2A_3]} = 0. \quad (4.4$$

The new products are immediately obtained from (4.45):

$$\mathcal{M}_{10}{}^{SE_1;A_1A_2E_2}M^C{}_{E_1E_2} = -\frac{2}{3}\mathcal{M}_{12}{}^{A_1A_2;SC}$$

$$\mathcal{M}_{10}{}^{S_1S_2;AE_1E_2}M^B{}_{E_1E_2} = \frac{2}{3}\hat{\mathcal{M}}_{12}{}^{S_1S_2,AB} = \frac{8}{9}\mathcal{M}_{12}^{S_1B;S_2A}$$

$$\mathcal{M}_{10}{}^{E_1E_2;A_1A_2A_3}M^B{}_{E_1E_2} = 0$$

$$\mathcal{M}_{10}{}^{SE;A_1A_2A_3}M^{B_1B_2}{}_E \;=\; -\frac{10}{9}(\mathcal{M}_{12}{}^{S;B_1B_2A_1A_2A_3}+\mathcal{M}_{12}^{B_1;B_2SA_1A_2A_3}) + \frac{2}{27}(8\eta^{A_1B_1}\mathcal{M}_{12}{}^{A_2A_3;B_2S}+\eta^{SA_1}\mathcal{M}_{12}{}^{A_2A_3;B_1B_2})$$

$$\begin{aligned}\mathcal{M}_{10}{}^{S_1S_2;A_1A_2E}M^{B_1B_2}{}_E \;&=\; -\frac{10}{9}\mathcal{M}_{12}{}^{S_1;S_2A_1A_2B_1B_2}\\ &+\frac{4}{7\times 81}\{58\eta^{A_1B_1}\mathcal{M}_{12}^{S_1B_2;S_2A_2}+31\eta^{S_1B_1}\mathcal{M}_{12}{}^{A_1A_2;S_2B_2}\\ &+11\eta^{S_1A_1}\mathcal{M}_{12}^{A_2S_2;B_1B_2}-2\eta^{S_1S_2}\mathcal{M}_{12}^{A_1A_2;B_1B_2}\}\end{aligned}$$

$$\begin{aligned}&\mathcal{M}_{10}{}^{S_1S_2;A_1A_2A_3}M^{B_1B_2B_3}=\hat{\mathcal{M}}_{10}^{S_1S_2[A_1;A_2A_3]}M^{B_1B_2B_3}\\ &=-\frac{15}{11}\Big\{\frac{6}{7}(\eta^{S_1B_1}\mathcal{M}_{12}^{S_2;A_1A_2A_3B_2B_3}+\eta^{S_1B_1}\mathcal{M}_{12}^{A_1;S_2A_2A_3B_2B_3}+\eta^{A_1B_1}\mathcal{M}_{12}^{S_1;S_2A_2A_3B_2B_3})\\ &-\frac{26}{63}\eta^{S_1A_1}\mathcal{M}_{12}^{S_2;A_2A_3B_1B_2B_3}-\frac{8}{63}\eta^{S_1A_1}\mathcal{M}_{12}^{A_2;S_2A_3B_1B_2B_3}-\frac{1}{7}\eta^{S_1S_2}\mathcal{M}_{12}^{A_1;A_2A_3B_1B_2B_3}\Big\}\\ &+\frac{2}{45}\eta^{S_1A_1}\eta^{S_2B_1}\mathcal{M}_{12}^{A_2A_3;B_2B_3}-\frac{32}{21\times 15}\eta^{S_1A_1}\eta^{A_2B_1}\mathcal{M}_{12}^{B_2B_3;A_3S_2}\\ &+\frac{12}{35}\eta^{S_1B_1}\eta^{A_1B_2}\mathcal{M}_{12}^{A_2A_3;B_3S_2}+\frac{8}{35}\eta^{A_1B_1}\eta^{A_2B_2}\mathcal{M}_{12}^{S_1A_3;S_2B_3}\\ &-\frac{1}{35}\eta^{S_1S_2}\eta^{A_1B_1}\mathcal{M}_{12}^{A_2A_3;B_2B_3}\end{aligned} \qquad (4.4$$

The second irreducible piece has 7 indices; we can extract a seven-index object by removing o of the factors from $\hat{\mathcal{M}}_{12}^{XABY;C_1C_2}$ to obtain the structure

$$\hat{\mathcal{M}}_{10}^{XYZ;A_1A_2B_1B_2}=M^{XED_1}M^{Y}{}_{D_1D_2}M^{ZD_2F}M^{A_1A_2}{}_{E}M_{F}{}^{B_1B_2}. \qquad (4.5$$

It is clear that

$$\hat{\mathcal{M}}_{10}^{[X}{}_{YZ;A_1A_2]}{}_{B_1B_2}=0 \qquad \hat{\mathcal{M}}_{10}^{XY[Z;A_1A_2}{}^{B_1B_2]}=0$$
$$\hat{\mathcal{M}}_{10}^{XYZ;A_1A_2B_1B_2}=-\hat{\mathcal{M}}_{10}^{ZYX;B_1B_2A_1A_2} \qquad (4.5$$

aside from the obvious antisymmetry in A_1, A_2 and B_1, B_2.

The object in (4.50) is not irreducible because it is not traceless. Its only non-vanishing trac are

$$\hat{\mathcal{M}}_{10\,E}{}^{YZ;A_1A_2\,EB}=-\hat{\mathcal{M}}_{10}^{YZB;A_1A_2}$$

$$\hat{\mathcal{M}}_{10}^{XY}{}_{E}{}^{;EA\,B_1B_2}=\hat{\mathcal{M}}_{10}^{YXA;B_1B_2}$$

$$\hat{\mathcal{M}}_{10}^{XYZ;}{}_{E}{}^{AEB}=-\mathcal{N}^{YXZBA}$$

$$\mathcal{N}^{YXZBA} =: M^{XD_1}{}_{D_2} M^{YD_2}{}_{D_3} M^{ZD_3}{}_{D_4} M^{BD_4}{}_{D_5} M^{AD_5}{}_{D_1} \qquad (4.5$$

For $\mathcal{N}^{XYZBA}$ we have

$$\mathcal{N}^{XYZBA} = \mathcal{N}^{AXYZB}$$

$$\mathcal{N}^{XYZBA} = -\mathcal{N}^{ABZYX} \qquad (4.5$$

as well as, by using (2.6) in the last two factors,

$$\mathcal{N}^{XYZAB} = \mathcal{N}^{XYZBA} + \hat{\mathcal{M}}_{10}^{XYZ;BA} \qquad (4.5$$

Iterating (4.54) and using (4.44) one can derive the decomposition

$$\mathcal{N}^{XYZAB} = -\frac{1}{2}\hat{\mathcal{M}}_{10}^{ZAB;XY} - \hat{\mathcal{M}}_{10}^{BA(X;Y)Z} + \frac{1}{2}\hat{\mathcal{M}}_{10}^{XYZ;BA} \qquad (4.5$$

and therefore

$$\begin{aligned}
\mathcal{N}^{[XY]ZAB} &= -\frac{1}{2}\hat{\mathcal{M}}_{10}^{ZAB;XY} \\
\mathcal{N}^{[X^Y Z]AB} &= -\hat{\mathcal{M}}_{10}^{AB[X;Z]Y} - \frac{1}{2}\hat{\mathcal{M}}_{10}^{ABY;XZ} \qquad (4.5
\end{aligned}$$

expressions that will be needed later.

In order to obtain the second irreducible piece of this θ^{10} sector we can just project (4.5 according to the pattern $\ydiagram{2,2,1,1,1}$. One obtains the structure

$$\begin{aligned}
\mathcal{M}_{10}^{CD;A_1...A_5} &= \hat{\mathcal{M}}_{10}^{CA_1D;A_2...A_5} \\
&= M^{CEG_1} M^{A_1}{}_{G_1G_2} M^{DG_2F} M^{A_2A_3}{}_E M_F{}^{A_4A_5}, \qquad (4.5
\end{aligned}$$

which is completely antisymmetric in $A_1, ..., A_5$ (we remind the reader of our letter conventio and by virtue of (4.51) it is also antisymmetric in C, D:

$$\mathcal{M}_{10}^{CD;A_1...A_5} = -\mathcal{M}_{10}^{DC;A_1...A_5} \qquad (4.5$$

Its tracelessness is immediate from (3.11), (2.6), (2.3) and (4.3), and it also satisfies

$$\mathcal{M}_{10}^{C[D;A_1...A_5]} = 0 \qquad \mathcal{M}_{10}^{[C_1C_2;B_1...B_4]A} = 0 \qquad (4.5$$

and it is self-dual:

$$\mathcal{M}_{10}^{C_1C_2;B_1...B_5} = \frac{1}{5!}\epsilon^{B_1...B_5D_1...D_5}\mathcal{M}_{10}{}^{C_1C_2;}{}_{D_1...D_5} \qquad (4.6$$

The list of decompositions is:

$$\mathcal{M}_{10}^{A_1A_2;B_1B_2E_1E_2E_3} M_{E_1E_2E_3} = \frac{2}{5}\mathcal{M}_{12}{}^{A_1A_2;B_1B_2}$$

$$\mathcal{M}_{10}^{E_1A;B_1B_2B_3E_2E_3}M_{E_1E_2E_3}=0 \qquad \mathcal{M}_{10}^{E_1E_2;B_1\dots B_4E_3}M_{E_1E_2E_3}=0$$
$$\mathcal{M}_{10}^{E_1A;B_1\dots B_4E_2}M^C{}_{E_1E_2}=\mathcal{M}_{12}^{[C;A]B_1\dots B_4} \qquad \mathcal{M}_{10}^{E_1E_2;B_1\dots B_5}M^C{}_{E_1E_2}=2\mathcal{M}_{12}^{C;B_1\dots B_5}$$

$$\mathcal{M}_{10}^{A_1A_2;B_1B_2B_3E_1E_2}M^C{}_{E_1E_2}=$$
$$\mathcal{M}_{12}^{[C;A_1A_2]B_1B_2B_3}+\frac{2}{45}\eta^{A_1B_1}\mathcal{M}_{12}^{A_2C;B_2B_3}+\frac{7}{45}\eta^{CB_1}\mathcal{M}_{12}^{A_1A_2;B_2B_3}$$

$$\mathcal{M}_{10}^{A_1A_2;B_1\dots B_4E}M^{C_1C_2}{}_E=\frac{1}{11}\Big[\frac{5}{2}\eta^{A_1C_1}\left(\mathcal{M}_{12}^{A_2;C_2B_1\dots B_4}-\mathcal{M}_{12}^{C_2;A_2B_1\dots B_4}\right)$$
$$+\frac{2}{3}\eta^{A_1B_1}\left(7\mathcal{M}_{12}^{C_1;C_2A_2B_2B_3B_4}+5\mathcal{M}_{12}^{A_2;C_1C_2B_2B_3B_4}\right)$$
$$+2\eta^{B_1C_1}\left(5\mathcal{M}_{12}^{A_1;A_2C_2B_2B_3B_4}+2\mathcal{M}_{12}^{C_2;A_1A_2B_2B_3B_4}\right)\Big]$$
$$+\frac{1}{9\times 25}\left(\eta^{A_1B_1}\eta^{A_2B_2}\mathcal{M}_{12}^{B_3B_4;C_1C_2}+21\eta^{B_1C_1}\eta^{B_2C_2}\mathcal{M}_{12}^{A_1A_2;B_3B_4}-14\eta^{A_1B_1}\eta^{C_1B_2}\mathcal{M}_{12}^{A_2C_2;B_3B_4}\right)$$

$$\mathcal{M}_{10}^{EA;B_1\dots B_5}M^{C_1C_2}{}_E = \frac{5}{11}\Big[\eta^{AC_1}\mathcal{M}_{12}^{C_2;B_1\dots B_5}+\frac{3}{2}\eta^{AB_1}\mathcal{M}_{12}^{C_1;C_2B_2\dots B_5}$$
$$+\frac{3}{2}\eta^{B_1C_1}\mathcal{M}_{12}^{C_2;AB_2\dots B_5}-\frac{5}{2}\eta^{B_1C_1}\mathcal{M}_{12}^{A;C_2B_2\dots B_5}\Big]$$

$$\mathcal{M}_{10}^{A_1A_2;B_1\dots B_5}M^{C_1C_2C_3}=$$
$$-\frac{1}{6!}\left(\frac{1}{10}\epsilon^{B_1\dots B_5A_1A_2C_3E_1E_2}\mathcal{M}_{12}^{C_1C_2;}{}_{E_1E_2}+\frac{1}{2}\epsilon^{B_1\dots B_5C_1C_2C_3E_1E_2}\mathcal{M}_{12}^{A_1A_2;}{}_{E_1E_2}\right.$$
$$\left.-\frac{3}{5}\epsilon^{B_1\dots B_5A_2C_2C_3E_1E_2}\mathcal{M}_{12}^{A_1C_1;}{}_{E_1E_2}\right)$$
$$+\frac{1}{11}[2\eta^{A_1C_1}\eta^{A_2C_2}\mathcal{M}_{12}^{C_3;B_1\dots B_5}+5\eta^{A_1B_1}\eta^{A_2B_2}\mathcal{M}_{12}^{C_1;C_2C_3B_3B_4B_5}$$
$$-\frac{15}{2}\eta^{A_1B_1}\eta^{A_2C_1}\mathcal{M}_{12}^{C_2;C_3B_2\dots B_5}-15\eta^{B_1A_1}\eta^{B_2C_1}\left(\mathcal{M}_{12}^{A_2;C_2C_3B_3B_4B_5}+\mathcal{M}_{12}^{C_2;C_3A_2B_3B_4B_5}\right)$$
$$+5\eta^{B_1C_1}\eta^{B_2C_1}\left(3\mathcal{M}_{12}^{A_1;A_2C_3B_3B_4B_5}+\mathcal{M}_{12}^{C_3;A_1A_2B_3B_4B_5}\right)$$
$$+\frac{5}{4}\eta^{C_1A_1}\eta^{C_2B_1}\left(5\mathcal{M}_{12}^{C_3;A_2B_2\dots B_5}-9\mathcal{M}_{12}^{A_2;C_3B_2\dots B_5}\right)]$$
$$+\frac{1}{12}\eta^{B_1C_1}\eta^{B_2C_2}\eta^{B_3C_3}\mathcal{M}_{12}^{A_1A_2;B_4B_5}+\frac{1}{10}\eta^{B_1A_1}\eta^{B_2C_1}\eta^{B_3C_2}\mathcal{M}_{12}^{A_2C_3;B_4B_5}$$
$$+\frac{1}{60}\eta^{B_1A_1}\eta^{B_2A_2}\eta^{B_3C_1}\mathcal{M}_{12}^{C_2C_3;B_4B_5} \tag{4.6}$$

$\underline{\theta^8}$.

At the beginning of this section we introduced one of the irreducible parts of the θ^8 secto
namely the totally symmetric tensor in (4.1):

$$\mathcal{M}_8^{S_1S_2S_3S_4} = M^{S_1}{}_E{}^F M^{S_2}{}_F{}^G M^{S_3}{}_G{}^H M^{S_4}{}_H{}^E. \qquad (4.6$$

Its products with $M^{A_1A_2A_3}$ are particularly easy to decompose using (4.43):

$$\mathcal{M}_8^{S_1S_2S_3E} M^{A_1A_2}{}_E = \hat{\mathcal{M}}_{10}^{S_1S_2S_3;A_1A_2}$$

$$\begin{aligned} \mathcal{M}_8^{S_1S_2S_3S_4} M^{A_1A_2A_3} &= \frac{7}{6}\eta^{S_1A_1}\hat{\mathcal{M}}_{10}^{S_2S_3S_4;A_2A_3} - \frac{1}{4}\eta^{S_1S_2}\hat{\mathcal{M}}_{10}^{S_3S_4A_1;A_2A_3} \\ &= \frac{21}{10}\eta^{S_1A_1}\mathcal{M}_{10}^{S_2S_3;S_4A_2A_3} - \frac{1}{4}\eta^{S_1S_2}\mathcal{M}_{10}^{S_3S_4;A_1A_2A_3} \qquad (4.6 \end{aligned}$$

This sector contains two additional irreducible pieces (see Table 1). In order to isolate the
first we remove one factor from $\hat{\mathcal{M}}_{10}^{S_1S_2S_3;A_1A_2}$ to get the structure

$$\hat{\mathcal{M}}_8^{XY\,A_1A_2\,B_1B_2} = M^{XED} M^Y{}_D{}^F M^{A_1A_2}{}_E M^{B_1B_2}{}_F \qquad (4.6$$

with the following properties

$$\hat{\mathcal{M}}_8^{XY\,A_1A_2\,B_1B_2} = \hat{\mathcal{M}}_8^{YX\,B_1B_2\,A_1A_2}$$

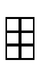
$$\hat{\mathcal{M}}_8^{[X^{Y}\,A_1A_2]\,B_1B_2} = 0 \qquad \hat{\mathcal{M}}_8^{X[Y\,^{A_1A_2}\,B_1B_2]} = 0 \qquad (4.6$$

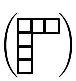
It is reducible,

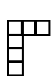
$$\hat{\mathcal{M}}_8^{XY\,EA}{}_E{}^B = -\mathcal{M}_8^{XYAB} \qquad (4.6$$

but easy to detrace:

$$\hat{\mathcal{M}}_8^{XY\,A_1A_2\,B_1B_2} = Traceless(\hat{\mathcal{M}}_8^{XY\,A_1A_2\,B_1B_2}) - \frac{1}{2}\eta^{A_1B_1}\mathcal{M}_8^{XYA_2B_2} \qquad (4.6$$

The traceless part is going to contain 2 irreducible pieces corresponding to the patterns 
and . First,

Y () $\hat{\mathcal{M}}_8^{XY\,A_1A_2\,B_1B_2} = \hat{\mathcal{M}}_8^{XY\,[A_1A_2\,B_1B_2]} + 2\hat{\mathcal{M}}_8^{A_1B_1\,[XA_2\,YB_2]}$ (4.6

and thus the  irreducible structure is

$$\mathcal{M}_8^{XY;B_1B_2B_3B_4} = \hat{\mathcal{M}}_8^{XY\,B_1B_2\,B_3B_4} \qquad (4.6$$

which is completely antisymmetric in $B_1, ..., B_4$ and, by (4.65), symmetric in X, Y:

$$\mathcal{M}_8^{XY;B_1B_2B_3B_4} = \mathcal{M}_8^{YX;B_1B_2B_3B_4} \qquad (4.7$$

and satisfying

$$\mathcal{M}_8^{X[Y;B_1B_2B_3B_4]} = 0. \qquad (4.7$$

Second,

$$\begin{aligned} Y\left(\boxplus\right)\hat{\mathcal{M}}_8^{XY\,A_1A_2\,B_1B_2} &= -\frac{1}{2}\left(\hat{\mathcal{M}}_8^{[X^{[A_1\,B_1B_2]}YA_2]} + \hat{\mathcal{M}}_8^{[X^{[A_1\,YA_2]}B_1B_2]}\right. \\ &+ \left.\hat{\mathcal{M}}_8^{[A_1^{[X\,B_1A_2]}YB_2]} + \hat{\mathcal{M}}_8^{[X^{[A_1\,B_1A_2]}YB_2]}\right) \qquad (4.7 \end{aligned}$$

giving as the $\boxplus$ irreducible structure the object

$$\mathcal{M}_8^{A_1A_2A_3;B_1B_2B_3} = \hat{\mathcal{M}}_8^{A_1B_1\,B_2B_3\,A_2A_3} \qquad (4.7$$

Of course it is completely antisymmetric in the A and B indices separately and, from (4.65 we see that it is symmetric upon interchange of both groups of indices

$$\mathcal{M}_8^{A_1A_2A_3;B_1B_2B_3} = \mathcal{M}_8^{B_1B_2B_3;A_1A_2A_3} \qquad (4.7$$

The remaining important property of this tensor can be derived from the definitions (4.73 (4.64) by using once more the properties of the θ^4 sector,

$$\mathcal{M}_8^{A_1A_2[C;B_1B_2B_3]} = 0 \qquad (4.7$$

that implies also

$$\mathcal{M}_8^{A[B_1B_2;B_3B_4]C} = 0 \qquad (4.7$$

By using the properties in (4.72) we can write finally for the decomposition in (4.67):

$$\begin{aligned} \hat{\mathcal{M}}_8^{XY\,A_1A_2\,B_1B_2} &= \mathcal{M}_8^{XY;A_1A_2B_1B_2} + 2\mathcal{M}_8^{A_1B_1;XA_2YB_2} \\ &+ \frac{3}{8}\left(3\mathcal{M}_8^{YA_1A_2;XB_1B_2} - \mathcal{M}_8^{XA_1A_2;YB_1B_2}\right) \\ &- \frac{1}{2}\eta^{A_1B_1}\mathcal{M}_8^{XYA_2B_2} \qquad (4.7 \end{aligned}$$

The lists of products decompositions for these irreducible pieces are

$$\mathcal{M}_{10}^{A_1A_2;B_1B_2E_1E_2E_3}M_{E_1E_2E_3} = \frac{2}{5}\mathcal{M}_{12}{}^{A_1A_2;B_1B_2}$$

$$\begin{aligned} \mathcal{M}_8^{S_1S_2;BE_1E_2E_3}M_{E_1E_2E_3} &= 0 \qquad & \mathcal{M}_8^{SE_1;B_1B_2E_2E_3}M_{E_1E_2E_3} &= 0 \\ \mathcal{M}_8^{SE_1;B_1B_2B_3E_2}M^C{}_{E_1E_2} &= -\frac{1}{2}\mathcal{M}_{10}^{SC;B_1B_2B_3} \qquad & \mathcal{M}_8^{E_1E_2;B_1\ldots B_4}M^C{}_{E_1E_2} &= 0 \end{aligned}$$

$$\begin{aligned}\mathcal{M}_8^{S_1S_2;B_1B_2E_1E_2}M^C{}_{E_1E_2} &= \frac{1}{3}\left(2\hat{\mathcal{M}}_{10}^{S_1S_2B_1;B_2C} - \hat{\mathcal{M}}_{10}^{S_1S_2C;B_1B_2}\right)\\ &= \frac{1}{5}\left(3\mathcal{M}_{10}^{S_1S_2;B_1B_2C} - \hat{\mathcal{M}}_{10}^{CS_1;S_2B_1B_2}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{M}_8^{S_1S_2;B_1B_2B_3E}M^{C_1C_2}{}_E &= \frac{10}{9}\mathcal{M}_{10}^{S_1C_1;S_2C_2B_1B_2B_3}\\ &+ \frac{1}{6}\left(\eta^{S_1C_1}\hat{\mathcal{M}}_{10}^{S_2C_2B_1;B_2B_3} + \eta^{S_1B_1}\hat{\mathcal{M}}_{10}^{S_2C_1B_2;B_3C_2}\right)\\ &+ \frac{1}{4}\eta^{B_1C_1}\left(-\frac{7}{3}\hat{\mathcal{M}}_{10}^{S_1S_2B_2;B_3C_2} + \hat{\mathcal{M}}_{10}^{S_1S_2C_2;B_2B_3}\right)\\ &= \frac{10}{9}\mathcal{M}_{10}^{S_1C_1;S_2C_2B_1B_2B_3}\\ &+ \frac{1}{6}\eta^{S_1C_1}\mathcal{M}_{10}^{S_2C_2;B_1B_2B_3} + \frac{1}{4}\eta^{S_1B_1}\mathcal{M}_{10}^{S_2C_1;B_2B_3C_2}\\ &+ \frac{1}{20}\eta^{B_1C_1}\left(-11\mathcal{M}_{10}^{S_1S_2;B_2B_3C_2} + 13\mathcal{M}_{10}^{S_1C_2;S_2B_2B_3}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{M}_8^{SE;B_1\ldots B_4}M^{C_1C_2}{}_E &= \frac{5}{9}\left(\mathcal{M}_{10}^{C_1C_2;SB_1\ldots B_4} + 5\mathcal{M}_{10}^{SC_1;C_2B_1\ldots B_4}\right)\\ &+ \frac{2}{3}\eta^{B_1C_1}\mathcal{M}_{10}^{SC_2;B_2B_3B_4}\end{aligned}$$

$$\begin{aligned}&\mathcal{M}_8^{S_1S_2;B_1\ldots B_4}M^{C_1C_2C_3} = -\frac{2}{3\times 5!}\epsilon^{B_1\ldots B_4^{[C_1}C_2C_3E_1E_2E_3}\mathcal{M}_{10}^{S_1]}{}_{S_2;}{}_{E_1E_2E_3}\\ &+\frac{4}{21}\Big[\eta^{S_1C_1}\left(10\mathcal{M}_{10}^{S_2C_2;C_3B_1\ldots B_4} + 2\mathcal{M}_{10}^{C_2C_3;S_2B_1\ldots B_4}\right)\\ &-\eta^{S_1B_1}\left(6\mathcal{M}_{10}^{S_2C_1;C_2C_3B_2B_3B_4} + \mathcal{M}_{10}^{C_1C_2;S_2C_3B_2B_3B_4}\right)\\ &-8\eta^{B_1C_1}\mathcal{M}_{10}^{S_1C_2;S_2C_3B_2B_3B_4} - \eta^{S_1S_2}\mathcal{M}_{10}^{C_1C_2;C_3B_1\ldots B_4}\Big]\\ &+\frac{1}{5}\Big[2\eta^{S_1C_1}\eta^{B_1C_2}\mathcal{M}_{10}^{S_2C_3;B_2B_3B_4} + 3\eta^{S_1B_1}\eta^{C_1B_2}\mathcal{M}_{10}^{S_2C_2;C_3B_3B_4}\\ &+3\eta^{B_1C_1}\eta^{B_2C_2}\left(\mathcal{M}_{10}^{S_1S_2;C_3B_3B_4} - \mathcal{M}_{10}^{S_1C_3;S_2B_3B_4}\right)\Big]\end{aligned} \qquad (4.7$$

and

$$\mathcal{M}_8^{A_1A_2A_3;E_1E_2E_3}M_{E_1E_2E_3} = 0 \qquad \mathcal{M}_8^{A_1A_2E_1;BE_2E_3}M_{E_1E_2E_3} = 0$$
$$\mathcal{M}_8^{A_1A_2A_3;BE_1E_2}M^C{}_{E_1E_2} = \frac{2}{3}\mathcal{M}_{10}^{BC;A_1A_2A_3} \qquad \mathcal{M}_8^{A_1A_2E_1;B_1B_2E_2}M^C{}_{E_1E_2} = -\frac{2}{3}\mathcal{M}_{10}^{CA_1;A_2B_1B_2}$$

$$\mathcal{M}_8^{A_1A_2E;B_1B_2B_3}M^{B_4B_5}{}_E = -\frac{2}{3}\mathcal{M}_{10}^{A_1A_2;B_1\ldots B_5}$$

$$
\begin{aligned}
&\mathcal{M}_8^{A_1A_2A_3;B_1B_2E} M^{C_1C_2}{}_E = \\
&\quad -\frac{8}{9}\left(\frac{2}{3}\mathcal{M}_{10}^{B_1B_2;A_1A_2A_3C_1C_2} + \mathcal{M}_{10}^{A_1A_2;A_3B_1B_2C_1C_2} - \frac{1}{6}\mathcal{M}_{10}^{C_1C_2;A_1A_2A_3B_1B_2}\right) \\
&\quad \frac{1}{15}\left[\frac{7}{3}\eta^{B_1C_1}\mathcal{M}_{10}^{B_2C_2;A_1A_2A_3} + \frac{7}{2}\eta^{A_1C_1}\mathcal{M}_{10}^{C_2A_2;A_3B_1B_2} - \frac{3}{2}\eta^{A_1B_1}\mathcal{M}_{10}^{A_2B_2;A_3C_1C_2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{M}_8^{A_1A_2A_3;B_1B_2B_3} M^{C_1C_2C_3} = \\
&+ \quad \frac{4}{15}\Big\{\eta^{B_3C_3}\left(-4\mathcal{M}_{10}^{B_1B_2;A_1A_2A_3C_1C_2} + \mathcal{M}_{10}^{C_1C_2;A_1A_2A_3B_1B_2} - 6\mathcal{M}_{10}^{A_1A_2;A_3B_1B_2C_1C_2}\right) \\
&\qquad +\eta^{A_3C_3}\left(-4\mathcal{M}_{10}^{A_1A_2;B_1B_2B_3C_1C_2} + \mathcal{M}_{10}^{C_1C_2;B_1B_2B_3A_1A_2} - 6\mathcal{M}_{10}^{B_1B_2;B_3A_1A_2C_1C_2}\right) \\
&\qquad +3\eta^{A_3B_3}\left(\mathcal{M}_{10}^{A_1A_2;B_1B_2C_1C_2C_3} + \mathcal{M}_{10}^{B_1B_2;A_1A_2C_1C_2C_3} - \mathcal{M}_{10}^{C_1C_2;C_3A_1A_2B_1B_2}\right)\Big\} \\
&+ \quad \frac{14}{75}[\eta^{B_1C_1}\eta^{B_2C_2}\mathcal{M}_{10}^{B_3C_3;A_1A_2A_3} + \eta^{A_1C_1}\eta^{A_2C_2}\mathcal{M}_{10}^{A_3C_3;B_1B_2B_3} \\
&\qquad -\frac{3}{2}\eta^{B_1C_1}\eta^{B_2A_1}\mathcal{M}_{10}^{B_3A_2;A_3C_2C_3} - \frac{3}{2}\eta^{A_1C_1}\eta^{A_2B_1}\mathcal{M}_{10}^{A_3B_2;B_3C_2C_3} \\
&\qquad -3\eta^{A_1C_1}\eta^{B_1C_2}\mathcal{M}_{10}^{C_3A_2;A_3B_2B_3} + \frac{1}{7}\eta^{A_1B_1}\eta^{A_2B_2}\mathcal{M}_{10}^{A_3B_3;C_1C_2C_3}]
\end{aligned}
\qquad (4.7
$$

$\underline{\theta^6}$.

In the decomposition (3.10) of the product of three $M^{A_1A_2A_3}$ we have two types of irreducib
structures:

$$\hat{\mathcal{M}}_6^{A\,B_1B_2\,C_1C_2} = M^{ADE}M^{B_1B_2}{}_D M^{C_1C_2}{}_E \qquad (4.8$$

and

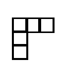
$$\mathcal{M}_6^{A_1A_2;B_1\ldots B_5} = M^{A_1A_2E}M_E{}^{B_1B_2}M^{B_3B_4B_5} \qquad (4.8$$

The expression (4.80) trivially satisfies

$$\hat{\mathcal{M}}_6^{A\,B_1B_2\,C_1C_2} = -\hat{\mathcal{M}}_6^{A\,C_1C_2\,B_1B_2} \qquad (4.8$$

and (2.6) implies

$$\hat{\mathcal{M}}_6^{[A\,B_1B_2]\,C_1C_2} = 0 \qquad (4.8$$

The tensor $\mathcal{M}_6^{[AB_1B_2]C_1C_2}$ must belong to the representation  and in order to make t
corresponding Young symmetry obvious, we define the new tensor

$$\mathcal{M}_6^{XY;B_1B_2B_3} = \hat{\mathcal{M}}_6^{X\,YB_1\,B_2B_3} \qquad (4.8$$

Both tensors are completely equivalent though, the inverse of (4.84) being

$$\hat{\mathcal{M}}_6^{A\,B_1B_2\,C_1C_2} = 3\mathcal{M}_6^{AB_1;B_2C_1C_2} \qquad (4.8$$

It is easy to see that $\mathcal{M}_6^{XY;B_1B_2B_3}$ must be symmetric in X,Y:

$$\mathcal{M}_6^{[XY];B_1B_2B_3} = M^{DE[X}M^{Y]B_1}{}_D M^{B_2B_3}{}_E = \\ -\frac{1}{2}M^{XY}{}_D M^{DE[B_1}M^{B_2B_3]}{}_E = 0 \qquad (4.8$$

where we have used (2.6) twice. Thus

$$\mathcal{M}_6^{XY;B_1B_2B_3} = \mathcal{M}_6^{YX;B_1B_2B_3} \qquad (4.8$$

The remaining important property of this tensor is

$$\mathcal{M}_6^{X[Y;B_1B_2B_3]} = 0 \qquad (4.8$$

as we have come to expect and can be immediately seen from (4.84) and (4.80). This time w
have the following product decompositions:

$$\mathcal{M}_6^{S_1S_2;E_1E_2E_3}M_{E_1E_2E_3} = 0 \qquad \mathcal{M}_6^{SE_1;BE_2E_3}M_{E_1E_2E_3} = 0$$
$$\mathcal{M}_6^{SE_1;B_1B_2E_2}M^C{}_{E_1E_2} = 0 \qquad \mathcal{M}_6^{S_1S_2;BE_1E_2}M^C{}_{E_1E_2} = -\frac{2}{3}\mathcal{M}_8^{S_1S_2BC}$$

$$\begin{aligned}\mathcal{M}_6^{S_1S_2;B_1B_2E}M^{C_1C_2}{}_E &= -\frac{4}{3}\mathcal{M}_8^{S_1C_1;S_2C_2B_1B_2} + \frac{2}{3}\mathcal{M}_8^{S_1S_2;B_1B_2C_1C_2} \\ &- \frac{1}{2}\mathcal{M}_8^{S_1B_1B_2;S_2C_1C_2} - \frac{1}{3}\eta^{B_1C_1}\mathcal{M}_8^{S_1S_2B_2C_2} \\ \mathcal{M}_6^{SE;B_1B_2B_3}M^{C_1C_2}{}_E &= \frac{4}{3}\mathcal{M}_8^{SC_1;C_2B_1B_2B_3} - \mathcal{M}_8^{SC_1C_2;B_1B_2B_3}\end{aligned}$$

$$\mathcal{M}_6^{S_1S_2;B_1B_2B_3}M^{B_4B_5B_6} = \frac{1}{2\times 5!}\epsilon^{B_1\ldots B_6E_1\ldots E_4}\mathcal{M}_8^{S_1S_2;}{}_{E_1\ldots E_4}$$

$$\begin{aligned}&\mathcal{M}_6^{S_1S_2;B_1B_2B_3}M^{C_1C_2C_3} = \\ &\quad \frac{3}{8\times 5!}\left(\epsilon^{B_1B_2B_3C_1C_2C_3E_1\ldots E_4}\mathcal{M}_8^{S_1S_2;}{}_{E_1\ldots E_4} + 2\epsilon^{S_1B_1B_2C_1C_2C_3E_1\ldots E_4}\mathcal{M}_8^{S_2B_3;}{}_{E_1\ldots E_4}\right) \\ &\quad +9\eta^{B_1C_1}\left(-\frac{1}{5}\mathcal{M}_8^{S_1C_2;S_2C_3B_2B_3} + \mathcal{M}_8^{S_1S_2;B_2B_3C_2C_3}\right) \\ &\quad +\frac{9}{10}\eta^{S_1C_1}\mathcal{M}_8^{S_2C_2;C_3B_1B_2B_3} + \frac{1}{2}\eta^{S_1B_1}\mathcal{M}_8^{S_2B_2;B_3C_1C_2C_3} \\ &\quad +\frac{3}{56}\left[-9\eta^{B_1C_1}\mathcal{M}_8^{S_1B_2B_3;S_2C_2C_3} - 12\eta^{S_1C_1}\mathcal{M}_8^{S_2C_2C_3;B_1B_2B_3}\right. \\ &\quad \left.+2\eta^{S_1B_1}\mathcal{M}_8^{S_2B_2B_3;C_1C_2C_3} + \eta^{S_1S_2}\mathcal{M}_8^{B_1B_2B_3;C_1C_2C_3}\right] \\ &\quad -\frac{3}{14}\eta^{B_1C_1}\eta^{B_2C_2}\mathcal{M}_8^{S_1S_2B_3C_3}\end{aligned} \qquad (4.8$$

Turning our attention to (4.81), we get the duality property

$$\mathcal{M}_6^{A_1A_2;B_1\ldots B_5} = -\frac{1}{5!}\epsilon^{B_1\ldots B_5 D_1\ldots D_5}\mathcal{M}_6{}^{A_1A_2;}{}_{D_1\ldots D_5} \qquad (4.9$$

as a direct consequence of the one for $\mathcal{M}_4^{C;D_1\ldots D_5}$ (eq. (3.8)). The following bracket property also immediate

$$\mathcal{M}_6^{A[C;B_1\ldots B_5]} = 0 \qquad (4.9$$

Finally, to complete this section we have the following list of decompositions:

$$\mathcal{M}_6^{A_1A_2;B_1B_2E_1E_2E_3}M_{E_1E_2E_3} = 0 \qquad \mathcal{M}_6^{AE_1;B_1B_2B_3E_2E_3}M_{E_1E_2E_3} = 0$$

$$\mathcal{M}_6^{E_1E_2;E_3B_1\ldots B_4}M_{E_1E_2E_3} = 0$$

$$\begin{aligned}
\mathcal{M}_6^{E_1E_2;B_1\ldots B_5}M^C{}_{E_1E_2} &= 0\\
\mathcal{M}_6^{AE_1;B_1\ldots B_4E_2}M^C{}_{E_1E_2} &= -\frac{3}{5}\mathcal{M}_8^{AC;B_1\ldots B_4}\\
\mathcal{M}_6^{A_1A_2;B_1B_2B_3E_1E_2}M^C{}_{E_1E_2} &= \frac{1}{5}\left(4\mathcal{M}_8^{CA_1;A_2B_1B_2B_3} - 3\mathcal{M}_8^{CA_1A_2;B_1B_2B_3}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_6^{B_1B_2;AB_3B_4B_5E}M^{CB_6}{}_E &= \frac{1}{20\times 5!}\epsilon^{B_1\ldots B_6E_1\ldots E_4}\mathcal{M}_8^{CA;}{}_{E_1\ldots E_4}\\
\mathcal{M}_6^{AB_1;B_2\ldots B_5E}M^{CB_6}{}_E &= -2\mathcal{M}_6^{B_1B_2;AB_3B_4B_5E}M^{CB_6}{}_E
\end{aligned}$$

$$\mathcal{M}_6^{AE;B_1\ldots B_5}M^{C_1C_2}{}_E = \frac{6}{5}\left(\frac{1}{5!}\epsilon^{B_1\ldots B_5C_1E_1\ldots E_4}\mathcal{M}_8^{AC_2;}{}_{E_1\ldots E_4} - \eta^{B_1C_1}\mathcal{M}_8^{AC_2;B_2\ldots B_5}\right)$$

$$\begin{aligned}
\mathcal{M}_6^{A_1A_2;B_1\ldots B_4E}M^{C_1C_2}{}_E = &-\frac{3}{5\times 5!}\epsilon^{B_1\ldots B_4A_1C_1E_1\ldots E_4}\mathcal{M}_8^{A_2C_2;}{}_{E_1\ldots E_4}\\
&+\frac{24}{25}\left[\eta^{B_1C_1}\mathcal{M}_8^{C_2A_1;A_2B_2B_3B_4} - \frac{1}{4}\eta^{A_1C_1}\mathcal{M}_8^{A_2C_2;B_1\ldots B_4} - \frac{1}{3}\eta^{A_1B_1}\mathcal{M}_8^{A_2C_1;C_2B_2B_3B_4}\right]\\
&-\frac{3}{10}\left(3\eta^{B_1C_1}\mathcal{M}_8^{C_2A_1A_2;B_2B_3B_4} + \eta^{A_1B_1}\mathcal{M}_8^{A_2C_1C_2;B_2B_3B_4}\right)
\end{aligned}$$

$$\mathcal{M}_6^{C_1C_2;B_1\ldots B_5}M^{C_3B_6B_7} = \frac{2}{7\times 5!}\epsilon^{B_1\ldots B_7E_1E_2E_3}\mathcal{M}_8^{C_1C_2C_3;}{}_{E_1E_2E_3}$$

$$
\begin{aligned}
&\mathcal{M}_6^{A_1A_2;B_1\ldots B_5} M^{C_1C_2C_3} = \\
= \quad & \frac{1}{32\times 35}(\epsilon^{B_1\ldots B_5A_1A_2E_1E_2E_3}\mathcal{M}_8^{C_1C_2C_3;}{}_{E_1E_2E_3} + 15\epsilon^{B_1\ldots B_5C_1C_2E_1E_2E_3}\mathcal{M}_8^{A_1A_2C_3;}{}_{E_1E_2E_3} \\
& -12\epsilon^{B_1\ldots B_5A_1C_1E_1E_2E_3}\mathcal{M}_8^{A_2C_2C_3;}{}_{E_1E_2E_3}) \\
& -\frac{1}{16\times 5}\left(\frac{4}{5}\eta^{A_1C_1}\epsilon^{B_1\ldots B_5C_2E_1\ldots E_4}\mathcal{M}_8^{A_2C_3;}{}_{E_1\ldots E_4} + \eta^{B_1C_1}\epsilon^{B_2\ldots B_5A_1C_2E_1\ldots E_4}\mathcal{M}_8^{A_2C_3;}{}_{E_1\ldots E_4}\right. \\
& \left.-\eta^{A_1B_1}\epsilon^{B_2\ldots B_5C_1C_2E_1\ldots E_4}\mathcal{M}_8^{A_2C_3;}{}_{E_1\ldots E_4}\right) \\
& +\frac{6}{5}(\eta^{B_1C_1}\eta^{B_2C_2}\mathcal{M}_8^{C_3A_1;A_2B_3B_4B_5} + \frac{3}{4}\eta^{A_1C_1}\eta^{B_1C_2}\mathcal{M}_8^{C_3A_2;B_2\ldots B_5} \\
& +\eta^{B_1A_1}\eta^{B_2C_1}\mathcal{M}_8^{A_2C_2;C_3B_3B_4B_5}) \\
& -\frac{3}{7}\left(\frac{15}{4}\eta^{B_1C_1}\eta^{B_2C_2}\mathcal{M}_8^{C_3A_1A_2;B_3B_4B_5} - 3\eta^{B_1A_1}\eta^{B_2C_1}\mathcal{M}_8^{A_2C_2C_3;B_3B_4B_5}\right. \\
& \left.+\frac{1}{4}\eta^{A_1B_1}\eta^{A_2B_2}\mathcal{M}_8^{B_3B_4B_5;C_1C_2C_3}\right) \qquad (4.9
\end{aligned}
$$

V θ^3-Fierz Identity and Γ-tracelessness.

The basic Fierz identity does not need to have four θ's but only three. Thus, (2.1) can be deriv
from

$$\theta^{(\pm)}\bar{\theta}^{\pm)}\mathcal{O}\theta^{(\pm)} = \frac{1}{96}\Pi^{(\pm)}\Gamma^{B_1B_2B_3}\mathcal{O}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{B_1B_2B_3}\theta^{(\pm)} \quad (5.$$

An immediate consequence of (5.1) is

$$\Gamma^{B_1B_2B3}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{B_1B_2B_3}\theta^{(\pm)} = 0 \quad (5.$$

and using (5.1) and (5.2) one easily obtains

$$\Gamma^{B_1B_2}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{B_1B_2A}\theta^{(\pm)} = 0 \quad (5.$$

Then one can finally Fierz the general uncontracted product to obtain

$$\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{A_1A_2A_3}\theta^{(\pm)} = \frac{1}{2}\Gamma_{A_1}\Gamma^{B}\theta^{(\pm)}\bar{\theta}^{(\pm)}\Gamma_{A_2A_3B}\theta^{(\pm)} \quad (5.$$

after using (5.1-5.3) and the properties of the Dirac algebra. Eq. (5.4) gives us the decompositi
of the product $\mathcal{M}^{A_1A_2A_3}\theta$ into irreducible pieces, and we see that the θ^3 irreducible spinor-tens
corresponding to $\left[\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}\right]$ is

$$\Theta_3^{A_1A_2} = \Gamma_E M^{A_1A_2E}\theta \quad (5.$$

which is obviously traceless and by (5.3) also Γ-traceless. Thus, (5.4) means

$$M^{A_1A_2A_3}\theta = \frac{1}{2}\Gamma^{A_1}\Theta_3^{A_2A_3} \quad (5.$$

Of course, this decomposition can be obtained easily by detracing and Young-projecting,

$$M^{A_1A_2A_3}\theta = Traceless(M^{A_1A_2A_3}\theta) + a\Gamma^{[A_1}\Gamma_E M^{A_2A_3]E}\theta \quad (5.$$

where "Traceless" now means both η- and Γ-traceless and there are no η terms on the r.h.s. b
cause the l.h.s. is trivially η-traceless. But the Traceless term in (5.7) vanishes because there a
no irreducible objects with 3 tensor indices in the θ^3 sector. The constant a is easily determin
by contracting (5.7) with Γ_{A_1}, to get $a = \frac{1}{2}$ and therefore reobtaining (5.6). The fermionic versi
of the Young-projector mentioned in the previous paragraph is straightforward enough, but
can become quite complicated for higher order decompositions. In order to simplify things, t
general way to proceed is as follows. First, we figure out the irreducible objects by contracti
as many indices as possible in the product $\mathcal{M}^{A_1A_2A_3}\Theta_n$ so that the number of remaining tens
indices are equal to the number of boxes of the corresponding Young-pattern, and then we app
the Young-projector to the resulting object. Next, we decompose the $\mathcal{M}_{n+1}\theta$ products in ter
of those irreducible pieces instead of decomposing $M^{A_1A_2A_3}\Theta_n$ since the former is much easi
than the latter in general. Finally, we may use the results of the bosonic decompositions
obtain the decomposition of $M^{A_1A_2A_3}\Theta_n$, since every fermionic irreducible object Θ_n is express
as some Γ-contraction of $\mathcal{M}_{n-1}\theta$. The procedure will be illustrated in the first few examples
the next section.

VI Irreducible Spinor-Tensors.

Unlike in the bosonic case, this time we will proceed forward.

$\underline{\theta^5}$.

It is easy to obtain the anti-selfdual spinor-tensor corresponding to $\left[\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{-3}{2}\right]$:

$$\begin{aligned}\Theta_5^{A_1\ldots A_5} &= M^{A_1A_2A_3}\Theta_3^{A_4A_5} = M^{A_1A_2A_3}M^{A_4A_5C}\Gamma_C\theta = \\ &= \Gamma_C\mathcal{M}_4^{C;A_1\ldots A_5}\theta \qquad (6.\end{aligned}$$

Evidently it is traceless, but it is also Γ-traceless:

$$\begin{aligned}\Gamma_D\Theta_5^{DA_1\ldots A_4} &= \frac{1}{5}\Gamma_D\left(3M^{DA_1A_2}M^{A_3A_4C} + 2M^{A_1A_2A_3}M^{A_4DC}\right)\Gamma_C\theta \\ &= \frac{3}{5}M^{DA_1A_2}M^{A_3A_4C}\Gamma_{DC}\theta = 0 \qquad (6.\end{aligned}$$

where we have used (5.3) as well as (2.6). The anti-selfduality

$$\Theta_5^{A_1\ldots A_5} = -\frac{1}{5!}\epsilon^{A_1\ldots A_5B_1\ldots B_5}\Theta_{5B_1\ldots B_5} \qquad (6.$$

together with (6.2) imply the property

$$\Gamma^{[B}\Theta_5^{A_1\ldots A_5]} = 0 \qquad (6.$$

The second irreducible θ^5 piece is:

$$\begin{aligned}\Theta_5^{A;B_1B_2} &= M^{B_1B_2}{}_E\Theta_3^{AE} = M^{B_1B_2}{}_EM^{AED}\Gamma_D\theta \\ &= \mathcal{M}_4^{DA;B_1B_2}\Gamma_D\theta \qquad (6.\end{aligned}$$

Usual tracelessness is also obvious here, while

$$\Gamma_D\Theta_5^{D;B_1B_2} = 0 \qquad (6.$$

follows again from (5.3). The other Γ-trace also vanishes:

$$\begin{aligned}\Gamma_D\Theta_5^{A;DB} &= \Gamma_DM^{DB}{}_EM^{AEF}\Gamma_F\theta = M^{DB}{}_EM^{AEF}\Gamma_{DF}\theta \\ &= M^{FD}{}_EM^{AEB}\Gamma_{DF}\theta = 0 \qquad (6.\end{aligned}$$

where we used our old friend (2.6) and (5.3) once more. Lastly, a property inherited fro $\mathcal{M}_4^{A_1A_2;B_1B_2}$ is

$$\Theta_5^{[A;B_1B_2]} = 0 \qquad (6.$$

Next we proceed to decompose products. By detracing one readily arrives at

$$\mathcal{M}_4^{A_1A_2;B_1B_2}\theta = \frac{1}{5}\left[\Gamma^{A_1}\Theta_5^{A_2;B_1B_2} + \Gamma^{B_1}\Theta_5^{B_2;A_1A_2}\right] \quad (6.$$

$$\mathcal{M}_4^{A;B_1\dots B_5}\theta = \frac{1}{10}\left(\Gamma^{A}\Theta_5^{B_1\dots B_5} + \Gamma^{B_1B_2B_3}\Theta_5^{A;B_4B_5}\right) \quad (6.1$$

With (6.9), (6.10) and (3.5) one can write the more general product

$$\begin{aligned} M^{A_1A_2A_3}M^{B_1B_2B_3}\theta &= \frac{1}{2}\Gamma^{A_1}\Theta_5^{A_2A_3B_1B_2B_3} \\ &+\frac{3}{20}\left[\Gamma^{A_1}\Gamma^{B_1B_2}\Theta_5^{B_3;A_2A_3} + \Gamma^{B_1}\Gamma^{A_1A_2}\Theta_5^{A_3;B_2B_3}\right] \end{aligned} \quad (6.1$$

from which in turn we get

$$\begin{aligned} M^{A_1A_2A_3}\Theta_3^{B_1B_2} &= \Theta_5^{A_1A_2A_3B_1B_2} - \frac{3}{10}\Gamma^{A_1}\Gamma^{B_1}\Theta_5^{B_2;A_2A_3} \\ &+ \frac{3}{10}\Gamma^{A_1A_2}\Theta_5^{A_3;B_1B_2} - \frac{6}{10}\eta^{A_1B_1}\Theta_5^{B_2;A_2A_3} \end{aligned} \quad (6.1$$

$\underline{\theta^7}$.

For the representation $\left[\frac{5}{2}\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{-1}{2}\right]$ we need an object with 4 tensor indices, so consider

$$\begin{aligned} \hat{\Theta}_7^{A_1A_2;B_1B_2} &= M^{A_1A_2}{}_E\Theta_5^{E;B_1B_2} = M^{A_1A_2}{}_D M^{B_1B_2}{}_E\Theta_3^{DE} = \\ &= \Gamma_C M^{A_1A_2}{}_E\mathcal{M}_4^{CE;B_1B_2}\theta = 3\Gamma_C\mathcal{M}_6^{CA_1;A_2B_1B_2}\theta \end{aligned} \quad (6.1$$

This object is evidently antisymmetric in A_1, A_2 and in B_1, B_2, but it is also antisymmetric upo
interchange of both sets of indices:

$$\hat{\Theta}_7^{A_1A_2;B_1B_2} = -\hat{\Theta}_7^{B_1B_2;A_1A_2} \quad (6.1$$

Normal tracelessness is obvious and Γ-tracelessness follows from that of $\Theta_5^{A;B_1B_2}$:

$$\Gamma_E\hat{\Theta}_7^{EA;B_1B_2} = \Gamma_E\hat{\Theta}_7^{B_1B_2;AE} = 0 \quad (6.1$$

Also, from the definition we extract the properties

$$\begin{aligned} \hat{\Theta}_7^{A[B;C]D} &= \hat{\Theta}_7^{D[B;C]A} \\ \hat{\Theta}_7^{[A_1A_2;B_1B_2]} &= 0 \end{aligned} \quad (6.1$$

Clearly, this object must be irreducible; however, the corresponding Young pattern symmetry
not manifest, so we define the new object

$$\Theta_7^{B;A_1A_2A_3} = \hat{\Theta}_7^{B[A_1;A_2A_3]} = \Gamma_C\mathcal{M}_6^{CB;A_1A_2A_3}\theta \quad (6.1$$

Eq. (6.16) implies

$$\Theta_7^{[B;A_1A_2A_3]} = 0 \tag{6.1}$$

Again, these two spinor-tensors are equivalent and the inverse of (6.17) is

$$\hat{\Theta}_7^{A_1A_2;B_1B_2} = -3\Theta_7^{[B_1;B_2]A_1A_2} \tag{6.1}$$

For the representation $\left[\frac{7}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right]$ we need an object with 3 tensor indices, so try

$$\begin{aligned}\Theta_7^{ABC} &= \Gamma_D M^{DA}{}_E \Theta_5^{C;BE} = \Gamma_D M^{DAE} M^B{}_{FE}\Theta_3^{FC} = \\ &= \Gamma_D \mathcal{M}_4^{DA;B}{}_F\, \Theta_3^{FC}\end{aligned} \tag{6.2}$$

From (6.20), (5.5), (4.80), (4.85) and the properties of $\mathcal{M}_6^{S_1S_2;B_1B_2B_3}$ one can also obtain

$$\Theta_7^{ABC} = -\frac{3}{2}\Gamma_{D_1D_2}\mathcal{M}_6^{B(A;C)D_1D_2}\theta \tag{6.2}$$

which shows that Θ_7^{ABC} is symmetric in A, C. In order to show that it is completely symmetri
we need to prove symmetry in A, B:

$$\begin{aligned}\Theta_7^{[AB]C} &= -\frac{1}{2}\Gamma_D M^{ABE} M^D{}_{FE}\Theta_3^{FC} \\ &= -\frac{1}{2}M^{AB}{}_E \Gamma_D \Theta_5^{C;DE} = 0\end{aligned} \tag{6.2}$$

Thus:

$$\Theta_7^{ABC} = \Theta_7^{BAC} = \Theta_7^{CBA} = \Theta_7^{ACB} \tag{6.2}$$

Next let us show that it vanishes upon contraction with Γ_A,

$$\begin{aligned}\Gamma_C\Theta_7^{ABC} &= \Gamma_C\Gamma_D M^{DAE} M^B{}_{FE}\Theta_3^{FC} \\ &= 2M_C{}^{AE} M^B{}_{FE}\Theta_3^{FC} = -M^{BAE} M_{FCE}\Theta_3^{FC} = 0\end{aligned} \tag{6.2}$$

as it is clear from (5.5) and (2.3).

Now we proceed to list the $\theta^6 \times \theta$ decompositions. First, by Young projection we get

$$\Gamma_{E_1E_2}\mathcal{M}_6^{S_1S_2;CE_1E_2}\theta = -\frac{2}{3}\Theta_7^{S_1S_2C} \tag{6.2}$$

which can also be obtained from (6.21) plus (6.23). For the remaining $\mathcal{M}_6^{S_1S_2;B_1B_2B_3}\theta$ produc
we have, together with (6.17),

$$\begin{aligned}\Gamma_{E_1E_2}\mathcal{M}_6^{SE_1;E_2B_1B_2}\theta &= 0 \\ \Gamma_E\mathcal{M}_6^{S_1S_2;C_1C_2E}\theta &= \frac{1}{2}\Theta_7^{S_1;S_2C_1C_2} + \frac{1}{6}\Gamma^{C_1}\Theta_7^{C_2S_1S_2} \\ \mathcal{M}_6^{S_1S_2;B_1B_2B_3}\theta &= \frac{1}{7}\Gamma^{S_1}\Theta_7^{S_2;B_1B_2B_3} + \frac{3}{28}\Gamma^{B_1}\Theta_7^{S_1;S_2B_2B_3} + \frac{1}{28}\Gamma^{B_1B_2}\Theta_7^{B_3S_1S_2}\end{aligned} \tag{6.2}$$

For $\mathcal{M}_6^{S_1S_2;B_1\ldots B_5}\theta$ we have instead:

$$\Gamma_{E_1\ldots E_4}\mathcal{M}_6^{A_1A_2;CE_1\ldots E_4}\theta = 0 \qquad \Gamma_{E_1\ldots E_4}\mathcal{M}_6^{AE_1;B_1B_2E_2E_3E_4}\theta = 0$$
$$\Gamma_{E_1E_2E_3}\mathcal{M}_6^{AE_1;E_2E_3B_1B_2B_3}\theta = -\frac{6}{5}\Theta_7^{A;B_1B_2B_3} \qquad \Gamma_{E_1E_2E_3}\mathcal{M}_6^{A_1A_2;B_1B_2E_1E_2E_3}\theta = -\frac{18}{5}\Theta_7^{A_1;A_2B_1B_2}$$

$$\Gamma_{E_1E_2}\mathcal{M}_6^{E_1E_2;B_1\ldots B_5}\theta = 0$$

$$\Gamma_{E_1E_2}\mathcal{M}_6^{AE_1;E_2B_1\ldots B_4}\theta = -\frac{6}{5}\Gamma^{B_1}\Theta_7^{A;B_2B_3B_4} \qquad \Gamma_{E_1E_2}\mathcal{M}_6^{A_1A_2;B_1B_2B_3E_1E_2}\theta = -\frac{9}{5}\Gamma^{B_1}\Theta_7^{A_1;A_2B_2B_3}$$

$$\begin{aligned}\Gamma_E\mathcal{M}_6^{AE;B_1\ldots B_5}\theta &= \Gamma^{B_1B_2}\Theta_7^{A;B_3B_4B_5}\\ \Gamma_E\mathcal{M}_6^{A_1A_2;B_1\ldots B_4E}\theta &= \frac{2}{35}[\Gamma^{A_1}\Gamma^{B_1}\Theta_7^{A_2;B_2B_3B_4} + 12\Gamma^{B_1B_2}\Theta_7^{A_1;A_2B_3B_4}\\ &\quad +2\eta^{A_1B_1}\Theta_7^{A_2;B_2B_3B_4}]\end{aligned}$$

$$\mathcal{M}_6^{A_1A_2;B_1\ldots B_5}\theta = -\frac{1}{7}\Gamma^{A_1}\Gamma^{B_1B_2}\Theta_7^{A_2;B_3B_4B_5} + \frac{3}{14}\Gamma^{B_1B_2B_3}\Theta_7^{A_1;A_2B_4B_5} \qquad (6.2$$

$\underline{\theta^9}$.

In this sector, we have the same representations than in the previous (θ^7) one. Inspired l (6.16), one defines

$$\begin{aligned}\Theta_9^{ABC} &= M^A{}_{DE}\hat{\Theta}_7^{BD;EC} = \frac{3}{2}\Gamma_F M^A{}_{DE}\mathcal{M}_6^{F(B;C)DE}\theta\\ &= -\Gamma_F\mathcal{M}_8^{FABC}\theta \qquad (6.2\end{aligned}$$

Its tracelessness and total symmetry have become obvious in the last equality in (6.28 hence this is the irreducible spinor-tensor corresponding to $\left[\frac{7}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}\right]$. By projecting the produ $M^{A_1A_2}{}_D\Theta_7^{S_1S_2D}$ one realizes that the other irreducible structure must be

$$\begin{aligned}\Theta_9^{B;A_1A_2A_3} &= M^{A_1A_2}{}_D\Theta_7^{A_3BD}\\ &= -\frac{3}{2}\Gamma_{E_1E_2}M^{A_1A_2}{}_D\mathcal{M}_6^{A_3D;BE_1E_2}\theta = \frac{3}{2}\Gamma_{E_1E_2}\mathcal{M}_8^{BE_1E_2;A_1A_2A_3}\theta. \qquad (6.29\end{aligned}$$

Exploiting the symmetry of Θ_7^{ABC} we can interchange the roles of A_3 and B in (6.29a) ar using the properties of $\mathcal{M}_8^{S_1S_2;D_1\ldots D_4}$ as well as the last equality in (6.29a), one can equally deri

$$\Theta_9^{B;A_1A_2A_3} = 2\Gamma_{EF}\mathcal{M}_8^{BE;FA_1A_2A_3}\theta. \qquad (6.29$$

The ordinary trace vanishes manifestly as does the first Γ-trace:

$$\Gamma_E \Theta_9^{E;A_1A_2A_3} = 0$$ (6.3

The other one also vanishes:

$$\begin{aligned}
\Gamma_E \Theta_9^{B;EA_1A_2} &= \frac{2}{3}\Gamma_E M^{EA_1}{}_D \Theta_7^{A_2BD} = \frac{2}{3}\Gamma_E M^{EA_1}{}_D \Gamma_F M^{FA_2}{}_C \Theta_5^{B;DC} = \\
&= \frac{2}{3} M^{EA_1}{}_D M_E{}^{A_2}{}_C \Theta_5^{B;DC} = -\frac{1}{3} M^{EA_2A_1} M_{EDC} \Theta_5^{B;DC} = 0
\end{aligned}$$ (6.3

as implied by (6.5) and (2.3). The remaining property inherited from (4.75) is

$$\Theta_9^{[B;A_1A_2A_3]} = 0$$ (6.3

Turning to the $\theta^8 \times \theta$ decompositions, the first one is trivially inferred from (6.28)

$$\mathcal{M}_8^{S_1S_2S_3S_4}\theta = -\frac{1}{4}\Gamma^{S_1}\Theta_9^{S_2S_3S_4}.$$ (6.3

From (6.29a) one successively derives the set:

$$\begin{aligned}
\Gamma_E \mathcal{M}_8^{A_1A_2A_3;B_1B_2E}\theta &= -\frac{2}{15}\Gamma^{B_1}\Theta_9^{B_2;A_1A_2A_3} + \frac{2}{10}\Gamma^{A_1}\Theta_9^{B_1;B_2A_2A_3} \\
\mathcal{M}_8^{A_1A_2A_3;B_1B_2B_3}\theta &= -\frac{1}{45}\left(\Gamma^{A_1A_2}\Theta_9^{A_3;B_1B_2B_3} + \Gamma^{B_1B_2}\Theta_9^{B_3;A_1A_2A_3}\right) \\
&+ \frac{1}{15}\Gamma^{A_1B_1}\Theta_9^{A_2;A_3B_2B_3}
\end{aligned}$$ (6.3

while from (6.29b) instead, the set

$$\begin{aligned}
\Gamma_E \mathcal{M}_8^{EA;B_1\ldots B_4}\theta &= -\frac{1}{2}\Gamma^{B_1}\Theta_9^{A;B_2B_3B_4} \\
\mathcal{M}_8^{AC;B_1\ldots B_4}\theta &= -\frac{2}{63}\left(\Gamma^A\Gamma^{B_1}\Theta_9^{C;B_2B_3B_4} + \Gamma^C\Gamma^{B_1}\Theta_9^{A;B_2B_3B_4}\right) \\
&+ \frac{1}{126}\left(\eta^{AB_1}\Theta_9^{C;B_2B_3B_4} + \eta^{CB_1}\Theta_9^{A;B_2B_3B_4}\right) \\
&+ \frac{1}{42}\Gamma^{B_1B_2}\left(\Theta_9^{A;CB_3B_4} + \Theta_9^{C;AB_3B_4}\right) + \frac{1}{42}\Gamma^{B_1B_2B_3}\Theta_9^{B_4AC}
\end{aligned}$$ (6.3

$\underline{\theta^{11}}$.

For the representation $\left[\frac{5}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right]$ we first construct the object with 3 indices by contractin $M^{A_1A_2A_3}$ with Θ_9^{ABC}. We define

$$\begin{aligned}
\hat{\Theta}_{11}^{A;BC} &= \Gamma_D M^{DAE}\Theta_9^{BC}{}_E = \Gamma_D \mathcal{M}_8^{DBCE}\Theta_3{}^A{}_E \\
&= \Gamma_{E_1E_2}\hat{\mathcal{M}}_{10}^{BCE_1;E_2A}\theta = \frac{3}{2}\Gamma_{E_1E_2}\mathcal{M}_{10}^{BC;AE_1E_2}\theta
\end{aligned}$$ (6.3

Then, we see that the tracelessness of $\hat{\Theta}_{11}^{A;BC}$ is trivially satisfied and the Γ-tracelessness is al
immediate from (6.36):

$$\Gamma_A \hat{\Theta}_{11}^{A;BC} = \Gamma_A \Gamma_D \mathcal{M}_8^{DBCE} {\Theta_3}^A{}_E = 2\eta_{AD} \mathcal{M}_8^{DBCE} {\Theta_3}^A{}_E = 0$$

$$\Gamma_B \hat{\Theta}_{11}^{A;BC} = \Gamma_B \Gamma_D M^{DAE} \Theta_9^{BC}{}_E = 2\eta_{BD} M^{DAE} \Theta_9^{BC}{}_E = 0 \qquad \text{(6.3}$$

So $\hat{\Theta}_{11}^{A;BC}$ is irreducible, and a useful property of $\hat{\Theta}_{11}^{A;BC}$ can be inferred from the group theor
i.e., we must have

$$\hat{\Theta}_{11}^{(A;BC)} = 0, \qquad \text{(6.3}$$

which reflects the fact that we can not have an irreducible object with totally symmetrized
indices in θ^{11}-sector (see Table1). In fact, (6.38) can be readily verified from the definition (6.36

$$\begin{aligned}
\hat{\Theta}_{11}^{(A;BC)} &= -\Gamma_{E_1E_2} \hat{\mathcal{M}}_{10}^{E_1(BC;A)E_2} \theta = \frac{1}{3} \Gamma_{E_1E_2} \hat{\mathcal{M}}_{10}^{BCA;E_1E_2} \theta \\
&= \frac{1}{3} \mathcal{M}_8^{BCAF} \Gamma_{E_1E_2} M^{E_1E_2}{}_F \theta = 0
\end{aligned}$$

Even though $\hat{\Theta}_{11}^{A;BC}$ is irreducible, its Young symmetry is not manifest, so we need to defi
a new object for $\left[\frac{5}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right]$:

$$\Theta_{11}^{B;CD} = \hat{\Theta}_{11}^{[C;D]B} = \frac{3}{2} \Gamma_{E_1E_2} \mathcal{M}_{10}^{BE_1;E_2CD} \theta. \qquad \text{(6.3}$$

Then, it is obvious from the definition (6.39) and (4.48) that $\Theta_{11}^{B;A_1A_2}$ satisfies

$$\Theta_{11}^{[B;A_1A_2]} = 0, \qquad \text{(6.4}$$

and the inverse of (6.39) is

$$\hat{\Theta}_{11}^{A;S_1S_2} = -\frac{4}{3} \Theta_{11}^{S_1;S_2A}. \qquad \text{(6.4}$$

Turning to the representation $\left[\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}\frac{3}{2}\right]$, we need an object with 5 totally antisymmetriz
tensor indices. Naturally, we define

$$\Theta_{11}^{A_1 \ldots A_5} = {M_E}^{A_1A_2} \Theta_9^{E;A_3A_4A_5} = -\Gamma_{E_1E_2} \mathcal{M}_{10}^{E_1E_2;A_1 \ldots A_5} \theta. \qquad \text{(6.4}$$

Again, the tracelessness is trivial, but for the Γ-tracelessness we need a little work:

$$\begin{aligned}
\Gamma_{A_1} \Theta_{11}^{A_1 \ldots A_5} &= \Gamma_{A_1} M^{E[A_1A_2} {\Theta_9}_E{}^{;A_3A_4A_5]} = \frac{2}{5} \Gamma_D M^{DA_2}{}_E \Theta_9^{E;A_3A_4A_5} \\
&= \frac{2}{5} M^{A_2A_3}{}_F \Gamma_D M^{DA_4}{}_E \Theta_7^{A_5FE} = \frac{3}{5} M^{A_2A_3}{}_F \Gamma_D \Theta_9^{F;DA_4A_5} = 0. \qquad \text{(6.4}
\end{aligned}$$

The irreducible object $\Theta_{11}^{A_1 \ldots A_5}$ satisfies similar properties to those of $\Theta_5^{A_1 \ldots A_5}$. First, it is self-du

$$\Theta_{11}^{A_1\ldots A_5} = \frac{1}{5!}\epsilon^{A_1\ldots A_5 B_1\ldots B_5}\Theta_{11\,B_1\ldots B_5} \qquad (6.4$$

and it satisfies

$$\Gamma^{[B}\Theta_{11}^{A_1\ldots A_5]} = 0. \qquad (6.4$$

While the self-duality (6.44) is obvious from (4.60) and (6.42), eq. (6.45) may be obtained fro (6.43) and (6.44) similarly to the case of $\Theta_5^{A_1\ldots A_5}$. In fact, the property (6.45) as well as (6.4) ma be also justified by the fact that: (1)$\Gamma^{[A_1}\Theta_{11}^{A_2\ldots A_6]}$ and $\Gamma^{[A_1}\Theta_5^{A_2\ldots A_6]}$ are irreducible and, (2)we ca not have an irreducible object with 6 fully antisymmetrized indices in the θ^{11}- and θ^5-sector $\Gamma^{[A_1}\Theta_{11}^{A_2\ldots A_6]}$ is indeed irreducible because it is both η- and Γ-traceless:

$$\Gamma_{A_1}\Gamma^{[A_1}\Theta_{11}^{A_2\ldots A_6]} = 0, \qquad (6.4$$

as can be seen by expanding the bracket.

Now let us list the $\theta^{10}\times\theta$ decompositions. For $\mathcal{M}_{10}^{S_1S_2;A_1A_2A_3}\theta$ products we first have (6.36 (6.39) and

$$\Gamma_{E_1E_2}\mathcal{M}_{10}^{S_1S_2;AE_1E_2}\theta = -\frac{8}{9}\Theta_{11}^{S_1;S_2A}. \qquad (6.4$$

Then from these two we successively obtain the remaining decompositions:

$$\Gamma_E\mathcal{M}_{10}^{EA;B_1B_2B_3}\theta = -\frac{1}{3}\Gamma^{B_1}\Theta_{11}^{A;B_2B_3}$$

$$\Gamma_E\mathcal{M}_{10}^{S_1S_2;A_1A_2E}\theta = \frac{4}{63}\Big(\Gamma^{S_1}\Theta_{11}^{S_2;A_1A_2} + 3\Gamma^{A_1}\Theta_{11}^{S_1;S_2A_2}\Big)$$

$$\mathcal{M}_{10}^{S_1S_2;A_1A_2A_3}\theta = \frac{1}{210}\Big(-9\Gamma^{S_1}\Gamma^{A_1}\Theta_{11}^{S_2;A_2A_3} + 4\eta^{S_1A_1}\Theta_{11}^{S_2;A_2A_3} + 6\Gamma^{A_1A_2}\Theta_{11}^{S_1;S_2A_3}\Big). \qquad (6.4$$

On the other hand, for $\mathcal{M}_{10}^{A_1A_2;B_1\ldots B_5}\theta$ we have (6.42) and

$$\Gamma_{E_1\ldots E_4}\mathcal{M}_{10}^{A_1A_2;BE_1\ldots E_4}\theta = -\frac{8}{5}\Theta_{11}^{B;A_1A_2}$$

$$\Gamma_{E_1\ldots E_4}\mathcal{M}_{10}^{BE_1;E_2E_3E_4A_1A_2}\theta = -\frac{2}{5}\Theta_{11}^{B;A_1A_2}$$

$$\Gamma_{E_1E_2E_3}\mathcal{M}_{10}^{A_1A_2;B_1B_2E_1E_2E_3}\theta = \frac{2}{5}\Gamma^{B_1}\Theta_{11}^{B_2;A_1A_2}$$

$$\Gamma_{E_1E_2E_3}\mathcal{M}_{10}^{AE_1;E_2E_3B_1B_2B_3}\theta = \frac{1}{5}\Gamma^{B_1}\Theta_{11}^{A;B_2B_3}$$

$$\Gamma_{E_1E_2E_3}\mathcal{M}_{10}^{E_1E_2;E_3A_1\dots A_4}\theta = 0$$

$$\Gamma_{E_1E_2}\mathcal{M}_{10}^{AE_1;E_2B_1\dots B_4}\theta = \frac{1}{5}\Theta_{11}^{AB_1\dots B_4} + \frac{4}{50}\Gamma^{B_1B_2}\Theta_{11}^{A;B_3B_4}$$

$$\begin{aligned}\Gamma_{E_1E_2}\mathcal{M}_{10}^{A_1A_2;B_1B_2B_3E_1E_2}\theta &= -\frac{1}{10}\Theta_{11}^{A_1A_2B_1B_2B_3} \\ &+ \frac{1}{200}\Big(17\Gamma^{B_1B_2}\Theta_{11}^{B_3;A_1A_2} - \Gamma^{A_1}\Gamma^{B_1}\Theta_{11}^{A_2;B_2B_3} - 4\eta^{A_1B_1}\Theta_{11}^{A_2;B_2B_3}\Big)\end{aligned}$$

$$\Gamma_{E}\mathcal{M}_{10}^{EA;B_1\dots B_5}\theta = \frac{1}{10}\Gamma^{A}\Theta_{11}^{B_1\dots B_5} + \frac{1}{30}\Gamma^{B_1B_2B_3}\Theta_{11}^{A;B_4B_5}$$

$$\begin{aligned}\Gamma_{E}\mathcal{M}_{10}^{A_1A_2;B_1\dots B_4E}\theta &= -\frac{1}{20}\Gamma^{A_1}\Theta_{11}^{A_2B_1\dots B_4} \\ &- \frac{1}{600}\Big(3\Gamma^{A_1}\Gamma^{B_1B_2}\Theta_{11}^{A_2;B_3B_4} + 11\Gamma^{B_1B_2B_3}\Theta_{11}^{B_4;A_1A_2} + 6\eta^{A_1B_1}\Gamma^{B_2}\Theta_{11}^{A_2;B_3B_4}\Big)\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{10}^{A_1A_2;B_1\dots B_5}\theta &= \frac{1}{88}\Big(\Gamma^{A_1A_2}\Theta_{11}^{B_1\dots B_5} - 2\eta^{A_1B_1}\Theta_{11}^{B_2\dots B_5A_2}\Big) \\ &+ \frac{1}{240}\Big(\Gamma^{A_1}\Gamma^{B_1B_2B_3}\Theta_{11}^{A_2;B_4B_5} - \Gamma^{B_1\dots B_4}\Theta_{11}^{B_5;A_1A_2}\Big) \qquad (6.4\end{aligned}$$

$\underline{\theta^{13}}$.

The only representation we have in this sector is just $\left[\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right]$ like in the θ^3-sector and th means that we need an object with 2 antisymmetric tensor indices again. Let us define

$$\Theta_{13}^{AB} = M^{A}{}_{E_1E_2}\Theta_{11}^{B;E_1E_2}. \qquad (6.5$$

Then the antisymmetry property of Θ_{13}^{AB} is automatically insured as soon as we obtain th following identity. That is, if we use (6.39), (4.48) and the first equation of (4.49), eq. (6.5 becomes

$$\begin{aligned}\Theta_{13}^{AB} &= \frac{3}{2}\Gamma_{D_1D_2}\mathcal{M}_{10}^{BD_1;D_2E_1E_2}M^{A}{}_{E_1E_2}\theta \\ &= -\frac{3}{2}\Gamma_{D_1D_2}\mathcal{M}_{10}^{BE_1;E_2D_1D_2}M^{A}{}_{E_1E_2}\theta \\ &= -\Gamma_{D_1D_2}\mathcal{M}_{12}^{D_1D_2;AB}\theta \qquad (6.5\end{aligned}$$

Further, the other expression for Θ_{13}^{AB} is also immediately obtained from (6.51) if we use the fir equation of (4.61), and (6.42):

$$\Theta_{13}^{AB} = -\frac{5}{2}\Gamma_{D_1D_2}\mathcal{M}_{10}^{D_1D_2;ABE_1E_2E_3}M_{E_1E_2E_3}\theta = \frac{5}{2}M_{E_1E_2E_3}\Theta_{11}^{E_1E_2E_3AB}. \qquad (6.5$$

On the other hand, the normal tracelessness of this antisymmetric spinor-tensor is trivial a

$$\Gamma_D\Theta_{13}{}^{DA} = 0 \qquad (6.5$$

is also obvious from the last equality in (6.52). So $\Theta_{13}^{A_1A_2}$ is the irreducible object correspondi to the representation $\left[\frac{3}{2}\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right]$. Now, for the $\theta^{12}\times\theta$ decompositions we have

$$\Gamma_E\mathcal{M}_{12}^{EA;B_1B_2}\theta = \frac{1}{12}\Big(\Gamma^A\Theta_{13}^{B_1B_2} - \Gamma^{B_1}\Theta_{13}^{B_2A}\Big)$$

$$\hat{\mathcal{M}}_{12}^{S_1S_2;X_1X_2}\theta = \frac{1}{33}\Gamma^{S_1X_1}\Theta_{13}^{S_2X_2}$$

$$\mathcal{M}_{12}^{A_1A_2;B_1B_2}\theta = \frac{1}{132}\Big(\Gamma^{A_1A_2}\Theta_{13}^{B_1B_2} + \Gamma^{B_1B_2}\Theta_{13}^{A_1A_2} + 2\Gamma^{A_1B_1}\Theta_{13}^{A_2B_2}\Big) \qquad (6.5$$

and

$$\Gamma_{E_1\ldots E_5}\mathcal{M}_{12}^{A;E_1\ldots E_5}\theta = 0$$

$$\Gamma_{E_1\ldots E_4}\mathcal{M}_{12}^{A;BE_1\ldots E_4}\theta = -\frac{4}{5}\Theta_{13}^{AB}$$

$$\Gamma_{E_1\ldots E_4}\mathcal{M}_{12}^{E_1;E_2E_3E_4A_1A_2}\theta = \frac{2}{5}\Theta_{13}^{A_1A_2}$$

$$\Gamma_{E_1E_2E_3}\mathcal{M}_{12}^{B;A_1A_2E_1E_2E_3}\theta = -\frac{3}{15}\Gamma^{A_1}\Theta_{13}^{A_2B}$$

$$\Gamma_{E_1E_2E_3}\mathcal{M}_{12}^{E_1;E_2E_3A_1A_2A_3}\theta = -\frac{1}{5}\Gamma^{A_1}\Theta_{13}^{A_2A_3}$$

$$\Gamma_{E_1E_2}\mathcal{M}_{12}^{B;A_1A_2A_3E_1E_2}\theta = -\frac{1}{450}\Big(19\Gamma^{A_1A_2}\Theta_{13}^{A_3B} + \Gamma^B\Gamma^{A_1}\Theta_{13}^{A_2A_3} + 4\eta^{BA_1}\Theta_{13}^{A_2A_3}\Big)$$

$$\Gamma_{E_1E_2}\mathcal{M}_{12}^{E_1;E_2A_1\ldots A_4}\theta = -\frac{2}{25}\Gamma^{A_1A_2}\Theta_{13}^{A_3A_4}$$

$$\Gamma_E\mathcal{M}_{12}^{E;A_1\ldots A_5}\theta = \frac{1}{30}\Gamma^{A_1A_2A_3}\Theta_{13}^{A_4A_5}$$

$$\Gamma_E \mathcal{M}_{12}^{B;A_1\dots A_4E}\theta = \frac{1}{450}\Big(4\Gamma^{A_1A_2A_3}\Theta_{13}^{A_4B} - \Gamma^B\Gamma^{A_1A_2}\Theta_{13}^{A_3A_4} - 2\eta^{BA_1}\Gamma^{A_2}\Theta_{13}^{A_3A_4}\Big)$$

$$\mathcal{M}_{12}^{B;A_1\dots A_5}\theta = \frac{1}{540}\Big(\Gamma^B\Gamma^{A_1A_2A_3}\Theta_{13}^{A_4A_5} + \Gamma^{A_1\dots A_4}\Theta_{13}^{A_5B}\Big) \qquad (6.5$$

$\underline{\theta^{15}}$.

Finally, for θ^{15}-sector we have again only one representation, which is $\left[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}\right]$ and t corresponding irreducible object is a spinor with no tensor indices just like θ, but with opposi chirality in this case. So the only possible candidate for Θ_{15} is:

$$\Theta_{15} \equiv \Theta = \Gamma_D M^{DE_1E_2}\Theta_{13E_1E_2} = \Gamma_{E_1E_2E_3}\mathcal{M}^{E_1E_2E_3}\theta. \qquad (6.5$$

For the decompositions we have

$$\begin{aligned}
\Gamma_{E_1E_2}\mathcal{M}^{E_1E_2A}\theta &= \frac{1}{10}\Gamma^A\Theta \\
\Gamma_E\mathcal{M}^{EA_1A_2}\theta &= -\frac{1}{90}\Gamma^{A_1A_2}\Theta \\
\mathcal{M}^{A_1A_2A_3}\theta &= -\frac{1}{720}\Gamma^{A_1A_2A_3}\Theta
\end{aligned} \qquad (6.5$$

VII Products of $M^{A_1A_2A_3}$ with Spinor-Tensors

In this section we list the products of $M^{A_1A_2A_3}$ with all the Θ_n of section VI, since they a another necessary ingredient in the development of the tensor calculus. Other more esoter product identities are given in Appendix B.

$\underline{\theta^5}$

$$\begin{aligned} M^{A_1A_2A_3}\Theta_3^{B_1B_2} &= \Theta_5^{A_1A_2A_3B_1B_2} \\ &- \frac{3}{10}\Gamma^{A_1}\Gamma^{B_1}\Theta_5^{B_2;A_2A_3} + \frac{3}{10}\Gamma^{A_1A_2}\Theta_5^{A_3;B_1B_2} - \frac{6}{10}\eta^{A_1B_1}\Theta_5^{B_2;A_2A_3} \qquad (7. \end{aligned}$$

$\underline{\theta^7}$

$$\begin{aligned} M^{A_1A_2A_3}\Theta_5^{B_1\ldots B_5} &= -\frac{1}{14}\Gamma^{B_1\ldots B_4}\Theta_7^{B_5;A_1A_2A_3} + \frac{15}{14}\Big(\Gamma^{A_1}\Gamma^{B_1B_2B_3} - 2\eta^{A_1B_1}\Gamma^{B_2B_3}\Big)\Theta_7^{B_4;B_5A_2A_3} \\ &- \frac{5}{7}\Big(\Gamma^{A_1A_2}\Gamma^{B_1B_2} + 2\eta^{A_1B_1}\Gamma^{A_2}\Gamma^{B_2} - 2\eta^{A_1B_1}\eta^{A_2B_2}\Big)\Theta_7^{A_3;B_3B_4B_5} \qquad (7. \end{aligned}$$

$$\begin{aligned} M^{A_1A_2A_3}\Theta_5^{C;B_1B_2} &= \frac{5}{14}\Big(-\Gamma^{C}\Gamma^{A_1} + 4\eta^{CA_1}\Big)\Theta_7^{A_2;A_3B_1B_2} + \frac{1}{21}\Big(\Gamma^{C}\Gamma^{B_1} - 4\eta^{CB_1}\Big)\Theta_7^{B_2;A_1A_2A_3} \\ &- \frac{5}{56}\Big(\Gamma^{A_1}\Gamma^{B_1} - 10\eta^{A_1B_1}\Big)\Theta_7^{B_2;CA_2A_3} - \frac{5}{56}\Big(5\Gamma^{A_1}\Gamma^{B_1} - 2\eta^{A_1B_1}\Big)\Theta_7^{C;B_2A_2A_3} \\ &+ \frac{5}{28}\Gamma^{A_1A_2}\Theta_7^{A_3;CB_1B_2} - \frac{15}{28}\Gamma^{A_1A_2}\Theta_7^{C;A_3B_1B_2} - \frac{1}{21}\Gamma^{B_1B_2}\Theta^{C;A_1A_2A_3} \\ &+ \frac{1}{28}\Big(\Gamma^{A_1A_2}\Gamma^{B_1} - 4\eta^{A_1B_1}\Gamma^{A_2}\Big)\Theta_7^{A_3B_2C} \qquad (7. \end{aligned}$$

$\underline{\theta^9}$

$$\begin{aligned} M^{A_1A_2A_3}\Theta_7^{B;C_1C_2C_3} &= -\frac{1}{60}\Gamma^{A_1A_2A_3}\Theta_9^{B;C_1C_2C_3} + \frac{1}{140}\Gamma^{C_1C_2C_3}\Theta_9^{B;A_1A_2A_3} \\ &+\frac{1}{140}\Big(\Gamma^{B}\Gamma^{C_1C_2} + \frac{2}{3}\eta^{BC_1}\Gamma^{C_2}\Big)\Theta_9^{C_3;A_1A_2A_3} + \frac{1}{10}\eta^{BA_1}\Gamma^{A_2}\Theta_9^{A_3;C_1C_2C_3} \\ &-\frac{9}{280}\Big(\Gamma^{A_1}\Gamma^{C_1C_2} + \frac{2}{3}\eta^{A_1C_1}\Gamma^{C_2}\Big)\Theta_9^{C_3;BA_2A_3} - \frac{1}{280}\Big(23\Gamma^{A_1}\Gamma^{C_1C_2} - 22\eta^{A_1C_1}\Gamma^{C_2}\Big)\Theta_9^{B;C_3A_2A_3} \\ &+\frac{3}{20}\Gamma^{A_1A_2}\Gamma^{C_1}\Theta_9^{B;A_3C_2C_3} - \frac{1}{20}\Big(\Gamma^{C_1}\Gamma^{A_1A_2} - 6\eta^{A_1C_1}\Gamma^{A_2}\Big)\Theta_9^{A_3;BC_2C_3} \\ &+\frac{1}{20}\Big(\Gamma^{B}\Gamma^{A_1}\Gamma^{C_1} + \eta^{A_1C_1}\Gamma^{B} + \eta^{BA_1}\Gamma^{C_1} - \eta^{BC_1}\Gamma^{A_1}\Big)\Theta_9^{A_2;A_3C_2C_3} \\ &-\frac{1}{140}\Big(\Gamma^{A_1A_2}\Gamma^{C_1C_2} - 6\eta^{A_1C_1}\Gamma^{A_2}\Gamma^{C_2} - 22\eta^{A_1C_1}\eta^{A_2C_2}\Big)\Theta_9^{A_3C_3B} \qquad (7. \end{aligned}$$

$$M^{A_1A_2A_3}\Theta_7^{S_1S_2S_3} = -\frac{1}{16}\Gamma^{A_1A_2A_3}\Theta_9^{S_1S_2S_3} + \frac{3}{112}\Big(5\Gamma^{A_1A_2}\Gamma^{S_1} - 8\eta^{S_1A_1}\Gamma^{A_2}\Big)\Theta_9^{A_3S_2S_3}$$
$$- \frac{1}{7}\eta^{S_1S_2}\Theta_9^{S_3;A_1A_2A_3} - \frac{3}{28}\Big(\Gamma^{S_1}\Gamma^{A_1} - 14\eta^{S_1A_1}\Big)\Theta_9^{S_2;S_3A_2A_3} \quad (7.$$

$\underline{\theta^{11}}$

$$M^{A_1A_2A_3}\Theta_9^{S_1S_2S_3} = \frac{2}{70}\Big(\Gamma^{A_1A_2}\Gamma^{S_1} - 10\eta^{S_1A_1}\Gamma^{A_2}\Big)\Theta_{11}^{S_2;S_3A_3}$$
$$- \frac{3}{70}\Big(\eta^{S_1S_2}\Gamma^{A_1} - 2\eta^{S_1A_1}\Gamma^{S_2}\Big)\Theta_{11}^{S_3;A_2A_3} \quad (7.$$

$$M^{A_1A_2A_3}\Theta_9^{C;B_1B_2B_3} = \frac{26}{330}\Gamma^{A_1A_2}\Theta_{11}^{A_3B_1B_2B_3C} + \frac{31}{330}\Gamma^{B_1B_2}\Theta_{11}^{B_3A_1A_2A_3C}$$
$$+\frac{1}{330}\Big(21\Gamma^C\Gamma^{A_1} + 109\eta^{CA_1}\Big)\Theta_{11}^{A_2A_3B_1B_2B_3} - \frac{91}{330}\eta^{CB_1}\Theta_{11}^{B_2B_3A_1A_2A_3} + \frac{213}{330}\eta^{A_1B_1}\Theta_{11}^{A_2A_3B_2B_3C}$$
$$+\frac{1}{1680}\Big(11\Gamma^{B_1B_2B_3}\Gamma^{A_1} - 50\eta^{A_1B_1}\Gamma^{B_2B_3}\Big)\Theta_{11}^{C;A_2A_3} - \frac{1}{40}\Gamma^{A_1A_2A_3}\Gamma^{B_1}\Theta_{11}^{C;B_2B_3}$$
$$+\frac{1}{120}\Big(2\Gamma^C\Gamma^{A_1A_2}\Gamma^{B_1} - 9\eta^{CA_1}\Gamma^{A_2}\Gamma^{B_1} - \eta^{CB_1}\Gamma^{A_1A_2} - \eta^{B_1A_1}\Gamma^C\Gamma^{A_2} - 18\eta^{CA_1}\eta^{A_2B_1}\Big)\Theta_{11}^{A_3;B_2B_3}$$
$$+\frac{1}{1680}\Big(-11\Gamma^{A_1}\Gamma^C\Gamma^{B_1B_2} - 30\eta^{CB_1}\Gamma^{B_2}\Gamma^{A_1} + 24\eta^{A_1B_1}\Gamma^C\Gamma^{B_2} + 140\eta^{CB_1}\eta^{B_2A_1}\Big)\Theta_{11}^{B_3;A_2A_3}$$
$$-\frac{1}{840}\Big(\Gamma^{B_1B_2}\Gamma^{A_1A_2} + 42\eta^{A_1B_1}\Gamma^{A_2}\Gamma^{B_2} + 168\eta^{A_1B_1}\eta^{A_2B_2}\Big)\Theta_{11}^{B_3;A_3C}$$
$$-\frac{1}{840}\Big(29\Gamma^{A_1A_2}\Gamma^{B_1B_2} + 22\eta^{A_1B_1}\Gamma^{A_2}\Gamma^{B_2} - 64\eta^{A_1B_1}\eta^{A_2B_2}\Big)\Theta_{11}^{C;A_3B_3} \quad (7.$$

$\underline{\theta^{13}}$

$$M^{A_1A_2A_3}\Theta_{11}^{C;B_1B_2} = \frac{1}{11}\Big[-\frac{1}{36}(\Gamma^C\Gamma^{A_1A_2A_3} - 12\eta^{CA_1}\Gamma^{A_2A_3})\Theta_{13}^{B_1B_2}$$
$$+ \frac{1}{60}(5\eta^{CB_1}\Gamma^{B_2}\Gamma^{A_1} - 18\eta^{CB_1}\eta^{B_2A_1} + 3\eta^{CA_1}\Gamma^{B_1B_2} + 3\eta^{A_1B_1}\Gamma^C\Gamma^{B_2})\Theta_{13}^{A_2A_3}$$
$$+ \frac{1}{60}(2\Gamma^C\Gamma^{B_1}\Gamma^{A_1A_2} - 8\eta^{CB_1}\Gamma^{A_1A_2} - 30\eta^{CA_1}\Gamma^{A_2}\Gamma^{B_1} - 20\eta^{CA_1}\eta^{A_2B_1})\Theta_{13}^{A_3B_2}$$
$$- \frac{1}{36}(\Gamma^{A_1A_2A_3}\Gamma^{B_1} + 6\eta^{B_1A_1}\Gamma^{A_2A_3})\Theta_{13}^{B_2C}$$
$$- \frac{1}{60}(2\Gamma^{B_1B_2}\Gamma^{A_1A_2} + 30\eta^{A_1B_1}\Gamma^{B_2}\Gamma^{A_2} - 80\eta^{A_1B_1}\eta^{A_2B_2})\Theta_{13}^{A_3C}\Big] \quad (7.$$

$$M^{A_1A_2A_3}\Theta_{11}^{B_1\ldots B_5} = \frac{1}{72}\Big(-\Gamma^{A_1A_2A_3}\Gamma^{B_1B_2B_3} + 6\eta^{B_1A_1}\Gamma^{A_2A_3}\Gamma^{B_2B_3} + 12\eta^{A_1B_1}\eta^{A_2B_2}\Gamma^{A_3}\Gamma^{B_3}\Big)\Theta_{13}^{B_4B}$$
$$+ \frac{1}{60}\Big(\Gamma^{A_1A_2}\Gamma^{B_1\ldots B_4} + 6\eta^{A_1B_1}\Gamma^{A_2}\Gamma^{B_2B_3B_4} - 8\eta^{A_1B_1}\eta^{A_2B_2}\Gamma^{B_3B_4}\Big)\Theta_{13}^{B_5A_3}$$
$$+ \frac{2}{720}\Big(\Gamma^{B_1\ldots B_5}\Gamma^{A_1} - 6\eta^{A_1B_1}\Gamma^{B_2\ldots B_5}\Big)\Theta_{13}^{A_2A_3} \quad (7.$$

$\underline{\theta^{15}}$

$$M^{A_1A_2A_3}\Theta_{13}^{B_1B_2} = -\frac{1}{7\times 720}\left(\Gamma^{A_1A_2A_3}\Gamma^{B_1B_2} + 6\eta^{B_1A_1}\Gamma^{A_2A_3}\Gamma^{B_2} - 24\eta^{B_1A_1}\eta^{B_2A_2}\Gamma^{A_3}\right)\Theta \quad (7.1$$

VIII Conclusions

We have presented here in detail the irreducible tensors and spinor-tensors contained in a scal superfield of definite chirality, $\Phi(x,\theta^{(+)})$ in particular but the results for $\Phi(x,\theta^{(-)})$ are trivial obtained making the changes explained in the introduction. The results for the most bas products of these irreducible structures have also been presented as a first step towards a fu tensor calculus. The remaining products can be derived by iteration of the formulae here ar will appear elsewhere.

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Appendix A. Conventions and Bosonic Identities

Our conventions are $\eta^{AB} = \eta_{AB} = diag(+-\ldots-)$, $\epsilon^{01\ldots9} = \epsilon_{01\ldots9} = 1$ and the Dirac algebra is

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \quad A, B = 0, 1, \ldots 9. \qquad \text{(A.}$$

Our definition for $\Gamma_{(11)}$ is

$$\Gamma_{(11)} = \Gamma_0\Gamma_1\ldots\Gamma_9$$

which satisfies

$$\Gamma^2_{(11)} = I \qquad \Gamma^{\dagger}_{(11)} = \Gamma_{(11)}$$

Then $\theta^{(+)} = \Pi^{(+)}\theta = \frac{1}{2}(I + \Gamma_{(11)})\theta$ belongs to the $[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}]$ representation of SO(10) whi $\theta^{(-)} = \Pi^{(-)}\theta = \frac{1}{2}(I - \Gamma_{(11)})\theta$ belongs to $[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{-1}{2}]$.

In 10 dimensions the Majorana and Weyl condition can be implemented simultaneously ar therefore our Majorana-Weyl spinors $\theta^{(\pm)}$ satisfy

$$\bar{\theta}^{(\pm)}\Gamma_{A_1\ldots A_n}\theta^{(\pm)} = 0 \qquad \text{for } n \neq 3, 7 \qquad \text{(A.}$$

The only independent bilinear in $\theta^{(\pm)}$ is then $M^{(\pm)}_{ABC} = \bar{\theta}^{(\pm)}\Gamma_{ABC}\theta^{(\pm)}$ since we have the identi (2.8). Powers of this bilinear satisfy many identities, implied by the basic Fierz one, that a used in the straightforward derivation of the decomposition of the θ^6 product in section III. He is a list,

$$M_{A_1A_2}{}^{C_1}M_{A_3}{}^{C_2C_3} = -\frac{1}{3}M_{A_1A_2A_3}M^{C_1C_2C_3} + \delta^{C_1}_{A_1}M_{A_2A_3E}M^{C_2C_3E}$$

$$\begin{aligned} M^{E[A_1A_2}M^{B_1B_2B_3]} &= \frac{2}{5}M^{EA_1A_2}M^{B_1B_2B_3} + \frac{3}{5}M^{EB_1B_2}M^{B_3A_1A_2} \\ &- \frac{3}{10}\eta^{EB_1}M^{A_1A_2D}M^{B_2B_3}{}_D - \frac{3}{5}\eta^{A_1B_1}M^{A_2ED}M^{B_2B_3}{}_D \end{aligned}$$

$$\begin{aligned} &M^{A_1C_1}{}_DM^{A_2C_2B_1}M^{B_2B_3D} = \\ &-\frac{1}{2}M_D{}^{C_1C_2}M^{A_1A_2B_1}M^{B_2B_3D} + \frac{1}{2}M_D{}^{C_1C_2}M^{A_1A_2D}M^{B_1B_2B_3} - \frac{1}{2}M^{B_1C_1C_2}M_D{}^{A_1A_2}M^{B_2B_3D} \\ &-\frac{1}{2}\eta^{A_1B_1}M^{A_2DE}M^{C_1C_2}{}_DM^{B_2B_3}{}_E - \eta^{A_1C_1}M^{C_2DE}M^{A_2B_1}{}_DM^{B_2B_3}{}_E - \frac{1}{2}\eta^{B_1C_1}M^{C_2DE}M^{A_1A_2}{}_D \end{aligned}$$

$$\begin{aligned} M^{A_3C_3B_3}M^{A_1A_2B_1}M^{C_1C_2B_2} &= \frac{1}{6}M^{B_1B_2B_3}M^{A_1A_2A_3}M^{C_1C_2C_3} - \frac{1}{2}M^{A_3B_1B_2}M^{B_3C_1C_2}M^{C_3A_1A_2} \\ &- \frac{1}{2}\eta^{B_1C_1}M^{A_1A_2A_3}M^{B_2B_3D}M^{C_2C_3}{}_D - \frac{1}{2}\eta^{A_1B_1}M^{A_2A_3}{}_DM^{B_2B_3D}M^{C_1} \\ &- \frac{1}{2}\eta^{A_1B_1}M^{A_2A_3}{}_DM^{B_2B_3C_1}M^{C_2C_3D} + \frac{1}{2}\eta^{A_1C_1}M^{A_2A_3}{}_DM^{B_2B_3D}M^{C_2} \\ &+ \frac{1}{2}\eta^{A_1B_1}M^{A_2A_3C_1}M^{B_2B_3}{}_DM^{C_2C_3D} + \eta^{B_1A_1}\eta^{B_2C_1}M^{B_3DE}M^{A_2A_3}{}_D \end{aligned}$$

$$\epsilon^{AB_4\ldots B_7CE_1\ldots E_4}M^{DB_1B_2}M_{DE_1E_2}M_{E_3E_4}{}^{B_3}=0$$

$$M^{DE[A}M_{DF_1F_2}M_{F_3E}{}^{C]}=0$$

$$M^{AB_1B_2}M^{B_3B_4B_5}M^{B_6B_7C}=-\frac{2}{7\times 5!}\epsilon^{B_1\ldots B_7E_1E_2E_3}M^{DAF}M_{DE_1E_2}M_{E_3F}{}^{C}$$

$$\begin{aligned} M^{AD_1D_2}M^{D_3D_4E}M^{C_1C_2}{}_{E} &= \frac{5}{6}M^{E[AD_1}M^{D_2D_3D_4]}M^{C_1C_2}{}_{E} \\ &+ \frac{5}{3}M^{[AD_1D_2}M^{C_1C_2]E}M^{D_3D_4}{}_{E}+\frac{2}{3}\eta^{D_4C_1}M^{AE_1D_1}M^{D_2D_3E_2}M_{E_1E_2}{}^{C_2} \end{aligned}$$

$$\begin{aligned} &\frac{1}{5!}\epsilon^{B_1\ldots B_6D_1\ldots D_4}M^{A}{}_{D_1D_2}M_{D_3D_4E}M^{C_1C_2E}= \\ &\quad\frac{2}{3}\frac{1}{5!}\epsilon^{B_1\ldots B_6D_1D_2D_3C_1}M^{AE_1}{}_{D_1}M_{D_2D_3}{}^{E_2}M_{E_1E_2}{}^{C_2} \\ &\quad+\frac{4}{3}\eta^{AB_1}M_{E}{}^{B_2B_3}M^{B_4B_5B_6}M^{C_1C_2E}+\frac{2}{3}\eta^{C_1B_1}M_{E}{}^{B_2B_3}M^{B_4B_5B_6}M^{C_2AE} \end{aligned}$$

A curious identity in the θ^{10} sector that is easy to prove is

$$\epsilon_{F_1F_2\ldots F_{10}}M^{AF_1F_2}M^{BF_3F_4}M^{CF_5F_6}M^{DF_7F_8}M^{EF_9F_{10}}=0$$

as it should be since no such symmetric object is allowed to exist.

Next we give a summary of how eq.(3.10) is derived directly from (2.12) or (3.3-3.5). W start with

$$\begin{aligned} &M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}= \\ &\quad=\left(-\frac{1}{4!}\epsilon^{B_1B_2B_3D_1\ldots D_5A_1A_2}M^{A_3}{}_{D_1D_2}M_{D_3D_4D_5}+\frac{3}{2}\eta^{A_1B_1}M^{A_2A_3}{}_{E}M^{B_2B_3E}\right)M^{C_1C_2C_3}= \\ &\quad=-\frac{1}{4!}\epsilon^{B_1B_2B_3D_1\ldots D_5A_1A_2}M^{A_3}{}_{D_1D_2}\left(-\frac{1}{4!}\epsilon_{D_3D_4D_5}{}^{E_1\ldots E_5C_1C_2}M^{C_3}{}_{E_1E_2}M_{E_3E_4E_5}+\frac{3}{2}\delta^{C_1}_{D_3}M_{D_4D_5E}M \right. \\ &\quad+\frac{3}{2}\eta^{A_1B_1}M^{A_2A_3}{}_{E}M^{B_2B_3E}M^{C_1C_2C_3} \end{aligned}$$

Expanding the product of the Levi-Civita symbols and using heavily the identities above, o gets after a lot algebra

$$M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3}=\frac{9}{8}M^{A_3C_1C_2}M^{B_3A_1A_2}M^{C_3B_1B_2}+I^{A_1A_2A_3B_1B_2B_3C_1C_2C_3}$$

with

$$
\begin{aligned}
&I^{A_1A_2A_3B_1B_2B_3C_1C_2C_3} = \\
&\quad = \frac{3}{8}\Big\{ 6\eta^{B_1C_1}M^{A_1A_2A_3}M^{B_2B_3D}M^{C_2C_3}{}_D + 4\eta^{A_1C_1}M^{A_2A_3D}M^{B_1B_2B_3}M^{C_2C_3}{}_D \\
&\quad + \frac{9}{2}\eta^{A_1B_1}M^{A_2A_3D}M^{B_2B_3}{}_D M^{C_1C_2C_3} - 6\eta^{A_1C_1}M^{A_2A_3B_1}M^{B_2B_3D}M^{C_2C_3}{}_D \\
&\quad + \frac{3}{2}\eta^{A_1B_1}\Big(M^{A_2A_3C_1}M^{B_2B_3D}M^{C_2C_3}{}_D - M^{A_2A_3D}M^{B_2B_3C_1}M^{C_2C_3}{}_D \Big) \\
&\quad - \frac{9}{2}\eta^{B_1C_1}M^{A_1A_2D}M^{B_2B_3}{}_D M^{A_3C_2C_3} - 3\eta^{A_1B_1}\eta^{A_2C_1}M^{A_3}{}_{DE}M^{B_2B_3D}M^{C_2C_3E} \\
&\quad - 6\eta^{A_1C_1}\eta^{B_1C_2}M^{C_3}{}_{DE}M^{A_2A_3D}M^{B_2B_3E} - 6\eta^{A_1B_1}\eta^{B_2C_1}M^{B_3}{}_{DE}M^{A_2A_3D}M^{C_2C_3E} \\
&\quad - 3\Big(\eta^{A_1C_1}\eta^{A_2C_2}M^{B_1B_2D}M^{B_3A_3E} + \eta^{B_1C_1}\eta^{B_2C_2}M^{A_1A_2D}M^{A_3B_3E} \Big) M^{C_3}{}_{DE} \Big\} \\
&\quad - \epsilon^{B_1B_2B_3D_1\ldots D_4C_1A_1A_2}M^{A_3}{}_{D_1D_2}M_{D_3D_4E}M^{C_2C_3E}
\end{aligned}
$$

Iterating this equation, we arrive at

$$
\begin{aligned}
&M^{A_3C_1C_2}M^{B_3A_1A_2}M^{C_3B_1B_2} - I^{C_1C_2A_3B_1B_2C_3A_1A_2B_3} = \frac{9}{8}M^{[A_3[A_1A_2}M^{B_3][B_1B_2}M^{C_3]^{C_1} \\
&\quad = \frac{5}{24}M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} \\
&\quad - \frac{1}{6}\Big(M^{A_3C_1C_2}M^{B_3A_1A_2}M^{C_3B_1B_2} + \frac{1}{2}M^{A_3B_1B_2}M^{B_3C_1C_2}M^{C_3A_1A_2} \Big) \\
&\quad + II^{A_1A_2A_3B_1B_2B_3C_1C_2C_3}
\end{aligned}
$$

with

$$
\begin{aligned}
&II^{A_1A_2A_3B_1B_2B_3C_1C_2C_3} = \\
&\quad - \frac{1}{4}\Big(\eta^{B_1C_1}M^{A_1A_2A_3}M^{B_2B_3D}M^{C_2C_3}{}_D + \eta^{A_1C_1}M^{A_2A_3D}M^{B_1B_2B_3}M^{C_2C_3}{}_D \\
&\quad + \eta^{A_1B_1}M^{A_2A_3D}M^{B_2B_3D}M^{C_1C_2C_3} \Big) \\
&\quad + \frac{1}{12}\Big(\eta^{A_1C_1}M^{A_2A_3B_1}M^{B_2B_3D}M^{C_2C_3}{}_D + \eta^{B_1C_1}M^{A_1A_2}{}_D M^{B_2B_3D}M^{C_2C_3A_3} \Big) \\
&\quad + \frac{1}{6}\Big(\eta^{A_1B_1}\eta^{A_2C_1}M^{A_3}{}_{DE}M^{B_2B_3D}M^{C_2C_3E} + \eta^{A_1C_1}\eta^{B_1C_2}M^{C_3}{}_{DE}M^{A_2A_3D}M^{B_2B_3E} \\
&\quad + \eta^{A_1B_1}\eta^{B_2C_1}M^{B_3}{}_{DE}M^{A_2A_3D}M^{C_2C_3E} \Big)
\end{aligned}
$$

Applying the (normalized) operator $\mathcal{S}(A,B,C)$ that fully symmetrizes upon interchange the letters A,B,C, to the equations we have just obtained, we get a system of two equatio with solution

$$
\begin{aligned}
M^{A_1A_2A_3}M^{B_1B_2B_3}M^{C_1C_2C_3} &= \frac{16}{65}\mathcal{S}(A,B,C)\Big[5I^{A_1A_2A_3B_1B_2B_3C_1C_2C_3} \\
&+ \frac{9}{2}\Big(I^{C_1C_2A_3B_1B_2C_3A_1A_2B_3} + II^{A_1A_2A_3B_1B_2B_3C_1C_2C_3} \Big) \Big]
\end{aligned}
$$

Let us now proceed to prove the duality properties of the tensors $\mathcal{M}_{12}^{A;B_1\ldots B_5}$ and $\mathcal{M}_{10}^{CD;B_1\ldots B}$
From (4.34) and (2.6) we can also write

$$\mathcal{M}_{12}^{B;A_1\ldots A_5} = \frac{1}{2} M^B{}_{D_1}{}^{D_2} M^F{}_{D_2}{}^{D_3} M^{A_1}{}_{D_3}{}^{D_4} M^{D_1}{}_{D_4}{}^{D_5} M^{A_2A_3}{}_{D_5} M_F{}^{A_4A_5} \tag{A.}$$

But:

$$\begin{aligned} & M^B{}_{D_1}{}^{D_2} M_{FD_2}{}^{D_3} M^{A_1}{}_{D_3}{}^{D_4} M^{D_1}{}_{D_4}{}^{D_5} M^{[A_2A_3}{}_{D_5} M^{FA_4A_5]} = \\ & \quad = M^B{}_{D_1}{}^{D_2} M_{FD_2}{}^{D_3} M^{A_1}{}_{D_3}{}^{D_4} M^{D_1}{}_{D_4}{}^{D_5} \frac{1}{5}\left(3M^{A_2A_3}{}_{D_5} M^{FA_4A_5} + 2M^{FA_2}{}_{D_5} M^{A_3A_4A_5}\right) \\ & \quad = \frac{3}{5} M^B{}_{D_1}{}^{D_2} M_{FD_2}{}^{D_3} M^{A_1}{}_{D_3}{}^{D_4} M^{D_1}{}_{D_4}{}^{D_5} M^{A_2A_3}{}_{D_5} M^{FA_4A_5}, \end{aligned}$$

so

$$\begin{aligned} \mathcal{M}_{12}^{B;A_1\ldots A_5} &= \frac{5}{6} M^B{}_{D_1}{}^{D_2} M_{FD_2}{}^{D_3} M^{A_1}{}_{D_3}{}^{D_4} M^{D_1}{}_{D_4}{}^{D_5} M^{[A_2A_3}{}_{D_5} M^{FA_4A_5]} \\ &= -\frac{5}{6}\frac{1}{5!} M^B{}_{D_1}{}^{D_2} M_{FD_2}{}^{D_3} M^{A_1}{}_{D_3}{}^{D_4} M^{D_1}{}_{D_4}{}^{D_5} \epsilon^{FA_2\ldots A_5E_1\ldots E_5} M_{D_5E_1E_2} M_{E_3E_4E_5} \end{aligned}$$

Now we have to "rotate" indices; that is, from the identity:

$$M_{FD_2}{}^{D_3} M^{D_1}{}_{D_4}{}^{D_5} M_{D_3}{}^{D_4[A_1} \epsilon^{FA_2\ldots A_5E_1\ldots E_5]} M_{D_5E_1E_2} M_{E_3E_4E_5} = 0 \tag{A.}$$

we see that

$$\begin{aligned} & M_{FD_2}{}^{D_3} M^{D_1}{}_{D_4}{}^{D_5} M_{D_5E_1E_2} M_{E_3E_4E_5} \Big[5M_{D_3}{}^{D_4A_1} \epsilon^{FA_2\ldots A_5E_1\ldots E_5} \\ & \quad - M_{D_3}{}^{D_4F} \epsilon^{A_1\ldots A_5E_1\ldots E_5} - 2M_{D_3}{}^{D_4E_1} \epsilon^{FA_1\ldots A_5E_2\ldots E_5} - 3M_{D_3}{}^{D_4E_3} \epsilon^{FA_1\ldots A_5E_1E_2E_4E_5} \Big] = 0, \end{aligned}$$

the second and third term vanish identically because of (2.3) and (3.11) respectively, and w
obtain

$$\begin{aligned} & M_{FD_2}{}^{D_3} M^{D_1}{}_{D_4}{}^{D_5} M_{D_5E_1E_2} M_{E_3E_4E_5} M_{D_3}{}^{D_4A_1} \epsilon^{FA_2\ldots A_5E_1\ldots E_5} = \\ & \quad = \frac{3}{5} M_{FD_2}{}^{D_3} M^{D_1}{}_{D_4}{}^{D_5} M_{D_5E_1E_2} M_{HE_3E_4} M_{D_3}{}^{D_4H} \epsilon^{FA_1\ldots A_5E_1\ldots E_4}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{M}_{12}^{B;A_1\ldots A_5} &= \frac{1}{2\times 5!} \epsilon^{A_1\ldots A_5FE_1\ldots E_4} M^B{}_{D_1}{}^{D_2} M^{D_1}{}_{D_4}{}^{D_5} M_{FD_2}{}^{D_3} M_{D_3}{}^{D_4H} M_{D_5E_1E_2} M_{HE_3E_4} \\ &= -\frac{1}{2\times 5!} \epsilon^{A_1\ldots A_5FE_1\ldots E_4} M^B{}_{D_1}{}^{D_2} M^{D_1}{}_{D_4}{}^{D_5} M^{D_4}{}_F{}^{D_3} M_{D_3D_2}{}^H M_{D_5E_1E_2} M_{HE_3E_4} \\ &= \frac{1}{2\times 5!} \epsilon^{A_1\ldots A_5E_1\ldots E_5} M^{BD_2}{}_{D_1} M^{D_5D_1}{}_{D_4} M_{E_1}{}^{D_4D_3} M_{D_2D_3}{}^H M_{D_5E_2E_3} M_{HE_4E_5} \\ &= \frac{1}{5!} \epsilon^{A_1\ldots A_5E_1\ldots E_5} \mathcal{M}_{12\,A_1\ldots A_5}^{B;}, \end{aligned}$$

the desired result. Notice the opposite sign with respect to the θ^4 piece, whose duality w
explicitly used. For $\mathcal{M}_{10}^{CD;B_1\ldots B_5}$ the derivation proceeds similarly and again one obtains a resu
opposite to the θ^4 one.

Appendix B. Fermionic Identities

In this Appendix we list identities involving some products of powers of M^{ABC} with the spino
tensors.

$$M_{E_1E_2}{}^{C}\Theta_5^{A_1A_2A_3E_1E_2} = \frac{3}{5}\Theta_7^{C;A_1A_2A_3}$$ (B.

$$\mathcal{M}_4^{EB;A_1A_2}\Theta_3{}^{C}{}_{E} = \frac{1}{2}\Big(\hat{\Theta}_7^{A_1A_2;BC} - \hat{\Theta}_7^{BA_1;A_2C}\Big) + \frac{1}{4}\Gamma^{A_1}\Theta_7^{A_2BC}$$ (B.

$$\begin{aligned} &M^{CE_1E_2}M^{A_1A_2}{}_{E_1}M^{B_1B_2}{}_{E_2}\theta = \\ &\quad = \frac{1}{28}\Big(\Gamma^{C}\hat{\Theta}_7^{A_1A_2;B_1B_2} + 2\Gamma^{A_1}\hat{\Theta}_7^{A_2B_1;B_2C} - 2\Gamma^{A_1}\hat{\Theta}_7^{A_2C;B_1B_2} - \Gamma^{A_1}\Gamma^{B_1}\Theta_7^{A_2B_2C} \\ &\quad -\Gamma^{C}\hat{\Theta}_7^{B_1B_2;A_1A_2} - 2\Gamma^{B_1}\hat{\Theta}_7^{B_2A_1;A_2C} + 2\Gamma^{B_1}\hat{\Theta}_7^{B_2C;A_1A_2} + \Gamma^{B_1}\Gamma^{A_1}\Theta_7^{B_2A_2C}\Big) \end{aligned}$$ (B.

$$\hat{\Theta}_7^{BA_1;A_2C} = \frac{3}{2}\Theta_7^{(B;C)A_1A_2}$$ (B.

$$\Gamma_E M^{EBF}\hat{\Theta}_{7F}{}^{C;A_1A_2} = M^{A_1A_2E}\Theta_7^{BC}{}_{E}$$ (B.

$$\Theta_9^{ABC} = \frac{1}{24}\Gamma_{E_1E_2E_3}M^{E_1E_2E_3}\Theta_7^{ABC}$$ (B.

$$M^{AD}{}_{E}M^{B}{}_{DF}\Theta_7^{CEF} = \hat{\Theta}_{11}^{C;AB}$$ (B.

$$M_{E_1E_2}{}^{A}\Theta_9^{B;CE_1E_2} = \frac{2}{3}\hat{\Theta}_{11}^{B;AC}$$ (B.

$$\Theta_9^{(C;D)A_1A_2} = \frac{2}{3}M^{EA_1A_2}\Theta_7^{CD}{}_{E} + \frac{1}{3}\Gamma^{A_1}\Theta_9^{A_2CD}$$ (B.

$$\begin{aligned} &M_{E_1}{}^{AE_2}M_{E_2}{}^{BE_3}M_{E_3}{}^{CE_4}M_{E_4}{}^{DE_5}\Theta_{3E_5}{}^{E_1} = \\ &\quad = \frac{1}{42}\Big(2\Gamma^{A}\hat{\Theta}_{11}^{B;CD} + \Gamma^{A}\hat{\Theta}_{11}^{C;BD} - 4\Gamma^{B}\hat{\Theta}_{11}^{A;CD} + \Gamma^{B}\hat{\Theta}_{11}^{C;AD} \\ &\quad -4\Gamma^{C}\hat{\Theta}_{11}^{A;BD} - 5\Gamma^{C}\hat{\Theta}_{11}^{B;AD} - \Gamma^{D}\hat{\Theta}_{11}^{B;AC} - 2\Gamma^{D}\hat{\Theta}_{11}^{C;AB}\Big) \end{aligned}$$ (B.1

$$\Gamma_E M^{EA}{}_{F}\hat{\Theta}_{11}^{F;BC} = -\frac{1}{3}\Gamma^{(B}\Theta_{13}^{C)A}$$ (B.1

$$\Gamma_E M^{EA}{}_{F}\hat{\Theta}_{11}^{B;CF} = \frac{1}{2}\Gamma^{A}\Theta_{11}^{BC} + \frac{1}{3}\Gamma^{B}\Theta_{11}^{CA} + \frac{1}{6}\Gamma^{C}\Theta_{11}^{AB}$$ (B.1

Appendix C. Young Projector Method

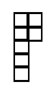
Let us consider a Young diagram R with n rows having m_i boxes in the i^{th} row ($m_1 \geq m_2$ $\ldots \geq m_n$) and having λ_j boxes in the j^{th} column ($n = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{m_1}$). The Youn projector corresponding to a particular (R_I) *standard tableau* [20] is given by

$$Y\,(R_I) = \alpha(R)\mathcal{Q}\mathcal{P} \qquad \text{(C.}$$
$$\mathcal{Q} = \prod_{i=1}^{m_1} Q_i \quad \mathcal{P} = \prod_{j=1}^{n} P_j$$

where P_j is the (normalized) operator that fully symmetrizes over the entries of the j^{th} row ar Q_i is the (normalized) one that fully antisymmetrizes over the entries of the i^{th} column. F operators so normalized, the normalization factor α needed for Y to be idempotent $Y^2 = Y$, i

$$\alpha(R) = \frac{dim(R)}{m!}\left(\prod_{j=1}^{n} m_j!\right)\left(\prod_{i=1}^{m_1} \lambda_i!\right) \qquad \text{(C.}$$

where $m = \sum_{j=1}^{n} m_j = \sum_{i=1}^{\lambda_i}$ is the total number of boxes in the Young diagram and $dim(R)$ the dimension of the irreducible representation of the symmetric group $\mathcal{S}_m$ corresponding to t diagram R [21]. The products of factorials in (C.2) appear because we considered normalized and P_j in (C.1) ($Q_i^2 = Q_i, P_j^2 = P_j$).

There are 14 standard tableaux associated with the diagram , however, due to identi (2.6) many of them do not contribute. The tableaux that give non-vanishing results are

$$\begin{array}{ll}1&4\\2&5\\3&\\6&\\7&\end{array}\quad\begin{array}{ll}1&4\\2&6\\3&\\5&\\7&\end{array}\quad\begin{array}{ll}1&4\\2&7\\3&\\5&\\6&\end{array}\quad\begin{array}{ll}1&5\\2&6\\3&\\4&\\7&\end{array}\quad\begin{array}{ll}1&5\\2&7\\3&\\4&\\6&\end{array}\quad\begin{array}{ll}1&6\\2&7\\3&\\4&\\5&\end{array} \qquad \text{(C.}$$

and the results for all the tableaux can be inferred from the first two

$$\begin{aligned} Y\begin{pmatrix}1&4\\2&5\\3&\\6&\\7&\end{pmatrix} & M^{A_1A_2A_3}M^{B_1B_2D}M^{C_1C_2}{}_D = \\ &= \frac{\alpha}{4}(M^{[A_1A_2A_3}M^{C_1C_2]}{}_D M^{B_1B_2D} + M^{[B_1A_2A_3}M^{C_1C_2]}{}_D M^{A_1B_2D} \\ &+ M^{[A_1B_2A_3}M^{C_1C_2]}{}_D M^{B_1A_2D} + M^{[B_1B_2A_3}M^{C_1C_2]}{}_D M^{A_1A_2D}) \end{aligned} \qquad \text{(C.}$$

$$Y\begin{pmatrix}1&4\\2&6\\3&\\5&\\7&\end{pmatrix} M^{A_1A_2A_3}M^{B_1B_2D}M^{C_1C_2}{}_D =$$

$$
\begin{aligned}
&= \frac{\alpha}{8}(M^{[A_1A_2A_3}M^{B_2C_2]}{}_D M^{C_1B_1D} + M^{[B_1A_2A_3}M^{B_2C_2]}{}_D M^{A_1C_1D} \\
&+ M^{[A_1C_1A_3}M^{B_2C_2]}{}_D M^{B_1A_2D} + M^{[B_1C_1A_3}M^{B_2C_2]}{}_D M^{A_1A_2D}), \qquad \text{(C.}
\end{aligned}
$$

the letter convention has been momentarily suspended in (C.4) and (C.5).

So, to obtain the total projection corresponding to the diagram $\yng(2,2,1,1,1)$ we add the contributio of all the standard tableaux in (C.3)

$$
\begin{aligned}
Y\left(\yng(2,2,1,1,1)\right) & M^{A_1A_2A_3}M^{B_1B_2D}M^{C_1C_2}{}_D = \\
&= \frac{\alpha}{4}(M^{[A_1A_2A_3}M^{B_1B_2]}{}_D M^{C_1C_2D} + M^{[A_1A_2A_3}M^{C_1C_2]}{}_D M^{B_1B_2D} \\
&+ 2M^{[A_1A_2A_3}M^{B_1C_1]}{}_D M^{B_2C_2D}) \qquad \text{(C.}
\end{aligned}
$$

$$
\alpha = \frac{dim\left(\yng(2,2,1,1,1)\right)}{7!}(2!2!)(5!2!) = \frac{8}{3}
$$

A comment is in order here. In projecting an arbitrary tensor one obtains a *different* i reducible representation for *each* standard tableau [20]. The same is not true here, of cours because of the nilpotency of the θ-tensors. Each irreducible representation appears only on at each level in Table 1. The number of degrees of freedom are dramatically reduced by t nilpotency of these structures and that is why the problem becomes manageable. For instanc the product $M^{A_1A_2A_3}M^{B_1B_2B_3}$ instead of having $\begin{pmatrix}10\\3\end{pmatrix} \times \begin{pmatrix}10\\3\end{pmatrix} = 120^2 = 14400$ degrees freedom, it has only $\begin{pmatrix}16\\4\end{pmatrix} = 770 + 1050 = 1820$. But doing the counting explicitly by subtrac ing the number of independent constraints implied by the conditions on the irreducible piec and otherwise derivable identities, can be an extremely painful task. However, one does not ne to dwell into all that detail, fortunately, but rather proceed to add all the projectors for t different standard tableaux corresponding to a Young diagram in order to consistently extra the *unique* representation involved in all the cases.

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